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Expected discounted penalty function of Markov-Modulated, Constant barrier and Stochastic income model

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Abstract: *This article considers a Markov -modulated risk process with stochastic premium income and a constant dividend barrier. We derive the integro-differential equations satisfied by the expected discounted penalty function regulated by an external environment. Applications of the integral equations are given to be the discounted expectation of the deficit at ruin. Explicit solution of the expected deficit at ruin for the model is obtained for exponential distribution. Finally in two state model, numerical example illustrates the effect of the different parameters.*
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I. Introduction

Expected discounted penalty function analyses the behaviour of the insurer's surplus. So it has always been of interest for years in mathematical insurance and many authors have been investigating on the different parameters of this function. The seminal paper by Gerber and Shiu in 1998 gave a detailed study of risk theory by not only simplifying but also giving a general approach to the treatment and the analysis of various risk-related quantities in to one single mathematical function - the expected discounted penalty function, or Gerber-Shiu function . The Markov- modulated risk model was first proposed by Reinhard [1] and Asmussen [2].Then later terms like expected discounted penalty function, constant barrier, and stochastic income and so on were added to this model to coincide with the real life. The Gerber-Shiu function is known to possess many properties, like the introduction of a dividend barrier strategy, it was shown by Lin et al. [4] and Gerber et al.[5] that the Gerber-Shiu function with a barrier can be expressed in terms of the Gerber-Shiu function without a barrier and the expected value of discounted dividend payments. This result was called dividends-penalty identity, and it holds true when the surplus process belongs to a class of Markov processes. Most of the researches have taken into account either Markov environment with constant barrier like in [10] or Markov environment with stochastic income as in [7],[8],[13],[18] .Since the risk management of an insurance company with constant premium fails to capture the uncertainty of the customer's arrival and the amount of premiums of different kinds of customers, models with stochastic income became more popular. To cast the cash inflow and outflow to be more realistic in insurance company, risk process with constant barrier and stochastic income has enhanced the flexibility of the model. Hua Dong et al.[9] has derived the integral equation of Gerber-shiu function of risk process with random income and a constant barrier. Wenguang Yu [19] considered the expected discounted penalty function of markov modulated risk process with stochastic premium income. Wenguang Yu and Yujuan Huang [15] expressed the integro-differential equations of the expected discounted penalty function of the markov modulated risk process with constant barrier under absolute ruin. However the combination of the Markov-modulated risk model with constant barrier and stochastic income is very rare. This motivates us to investigate such risk model in this work.

The rest of the paper is structured as follows. In section 2, the risk model under consideration is introduced in detail.In section 3, given an initial surplus and the initial environment state, the integroodifferential equation for the expected discounted penalty function is derived. In section 4, for exponential distribution , we obtain the expected discounted penalty function. The results are illustrated by numerical examples in section 5.Section 6 concludes the paper.

II. The Risk Model

In this paper, we let a complete probability space satisfying usual conditions containing all random variables and stochastic processes in our discussion.
In this model let the surplus process $\mathrm{U}(\mathrm{t})$ of an insurance company be defined as follows:

$$\mathrm{U}(\mathrm{t}) = \mathrm{u} + \mathrm{S}_2(\mathrm{t}) - \mathrm{S}_1(\mathrm{t}), \mathrm{t} \geq 0$$

Where $u \geq 0$ is the initial capital, $S_1(t) = \sum_{i=1}^{N(t)} X_i$,X_i is the ith claim amount and $S_2(t) = \sum_{i=1}^{M(t)} Y_i$, Y_i is the ith premium amount . $\{N(t); t \geq 0\}$ represents the number of claims occurring in $(0, t]$, and $\{M(t); t \geq 0\}$ represents the number of premium arrivals up to time t. Furthermore we have $\{J(t), t \geq 0\}$, an external environment process which is a homogenous irreducible and recurrent markov process with a finite state space $\{E = 1,2,3 \ldots .m\}$ with intensity matrix $\Delta = \{\alpha_{ij}\}_{i,j=1}^{m}$ where $\alpha_{ij} = -\alpha_i$ for $\in E$. As pointed by Asmussen[1], in health insurance this could be some type of epidemics or, in automobile insurance, this could also be certain type of climatic change.

The processes $\{N(t), M(t); t \geq 0\}$ are governed by the external environment $\{J(t), t \geq 0\}$. If $J(s) = i \; \forall \; s$ in a small interval $(t, t+h]$ then the number of claims occurring in that interval $N(t+h) - N(t)$ is assumed to follow a Poisson distribution with parameter $\lambda > 0$ and the ith claim amount X_i have the distribution $F(X)$ with density function $f_i(x)$ and finite mean m_X. Similarly the number of premium arrivals $\{M(t); t \geq 0\}$ has the Poisson distribution with parameter $\mu > 0$, and the corresponding premiums have distribution function $G(Y)$ with density function $g_i(y)$ and finite mean m_Y. The safety loading condition holds $\left[\sum_{i=1}^{M(t)} Y_i\right] > E\left[\sum_{i=1}^{N(t)} X_i\right]$. We assume the process $\{J(t), N(t), M(t); t \geq 0\}$ are mutually independent and has independent increments.

A barrier strategy considered in this paper assumes a horizontal barrier of level $b > 0$ such that when ever the surplus exceeds the level b, the excess is paid out immediately as dividend. Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the barrier strategy.

Let $\varphi_i(u, b)$ denote the expected discounted penality function with initial surplus amount to be u with a constant barrier b. This is defined as follows:

$$\varphi_i(u,b) = E[e^{-\delta T_b} w(U_b(T_b{}^-), |U_b(T_b)|) I(T_b < \infty) | U_b(0) = u],$$

Where $I(.)$ is the indicator function, $T_b = \inf\{t: U_b(t) < 0\}$ denotes the time of ruin, $U_b(T_b{}^-)$ is the surplus prior to ruin, $|U_b(T_b)|$ is the deficit at ruin, and $w(x, y)$ is a non negative bounded function on $[0, \infty) \times [0, \infty)$. We can interpret $e^{-\delta T_b}$ as the discount factor. The expected deficit at ruin for $U_b(0) = u$ is

$$\varphi_i(u,b) = E[e^{-\delta T_b} |U_b(T_b)| I(T_b < \infty) | U_b(0) = u]$$

III. The Integral Equation

In this section, we derive the integral equation for the expected discounted penalty function.

Theorem 1. For a small time interval $[0, t], t > 0$, $\varphi_i(u, b), i = 1,2 \ldots .m$ continuous with respect to u on $[0, \infty)$ and $w(x, y)$ is continuous with respect to x. We have the following integral equations

In $0 \leq u < b$

$$\begin{aligned}(\alpha_i + \lambda_i + \mu_i + \delta)\, \varphi_i(u,b) &= \lambda_i \left[\int_0^u \varphi_i(u-x,b)dF_i(x) + \int_u^\infty w(u, x-u)dF_i(x)\right] \\ &+ \mu_i \left[\int_0^{b-u} \varphi_i(u+y,b)dG_i(y) + \varphi_i(b,b)\int_{b-u}^\infty dG_i(y)\right] + \sum_{k=1,k\neq i}^{m} \alpha_{ik}\varphi_k(u,b) - (1)\end{aligned}$$

At $u = b$

$$(\alpha_i + \lambda_i + \delta)\varphi_i(b,b) = \lambda_i \left[\int_0^b \varphi_i(b-x,b)dF_i(x) + \int_b^\infty w(b, x-b)dF_i(x)\right] + \sum_{k=1,k\neq i}^{m} \alpha_{ik}\varphi_k(b,b) - (2)$$

Where $F_i(x)$ and $G_i(y)$ are the distribution of the number of claims and the premiums amounts, respectively.

Proof. Consider $U_b(t)$ in the small interval $[0, t], t > 0$ .The five different possible cases are as follows:

1. No claim occurs in $[0, t]$, no premium arrival in $[0, t]$ and no change in the external environment in $[0, t]$.
2. One claim occurs in $[0, t]$, no premium arrival in $[0, t]$ and no change in the external environment in $[0, t]$.
3. No claim occurs in $[0, t]$, one premium arrival in $[0, t]$ and no change in the external environment in $[0, t]$.
4. No claim occurs in $[0, t]$, no premium arrival in $[0, t]$ and a change in the external environment in $[0, t]$.
5. All other events with total probability $o(t)$.

By conditioning on the occurrence of claims, the occurrence of the premiums and the change in the external environment in $[0, t]$, the expected discounted penalty function $\varphi_i(u, b)$ is as follows:

$$\varphi_i(u,b) = \sum_{k=1}^{4} E_k[e^{-\delta T_b} w(U_b(T_b^{\,-}), |U_b(T_b)|) I(T_b < \infty) | U_b(0) = u] + o(t)$$
$$= (1 - \alpha_i t - \lambda_i t - \mu_i t) e^{-\delta t} \varphi_i(u,b) + \lambda_i t e^{-\delta t} \left[\int_0^u \varphi_i(u-x,b) dF_i(x) + \int_u^\infty w(u, x-u) dF_i(x) \right]$$
$$+ \mu_i t e^{-\delta t} \left[\int_0^{b-u} \varphi_i(u+y,b) dG_i(y) + \varphi_i(b,b) \int_{b-u}^\infty dG_i(y) \right] + t e^{-\delta t} \sum_{k=1, k\neq i}^{m} \alpha_{ik} \varphi_k(u,b)$$
$$+ o(t)$$

Expanding $e^{-\delta t}$, dividing by t and taking limit as $t \longrightarrow \infty$ we get (1)
Similarly, for $u = b$

$$\varphi_i(b,b) = (1 - \alpha_i t - \lambda_i t) e^{-\delta t} \varphi_i(b,b) + \lambda_i t e^{-\delta t} \left[\int_0^b \varphi_i(b-x,b) dF_i(x) + \int_b^\infty w(b, x-b) dF_i(x) \right]$$
$$+ t e^{-\delta t} \sum_{k=1, k\neq i}^{m} \alpha_{ik} \varphi_k(b,b)$$

Reducing and rearranging the above gives (2).

Remark 1. When $\delta > 0$ and $w(x,y) = 1$, the expected discounted penalty function simplifies to Laplace transform of the time to ruin, $\varphi_i(u,b) = E[e^{-\delta T_b} I(T_b < \infty) | U_b(0) = u]$ and (1), (2) simplifies to

$$(\alpha_i + \lambda_i + \mu_i + \delta)\, \varphi_i(u,b)$$
$$= \lambda_i \left[\int_0^u \varphi_i(u-x,b) dF_i(x) + 1 - F_i(x) \right]$$
$$+ \mu_i \left[\int_0^{b-u} \varphi_i(u+y,b) dG_i(y) + \varphi_i(b,b) \{1 - G_i(b-u)\} \right] + \sum_{k=1, k\neq i}^{m} \alpha_{ik} \varphi_k(u,b)$$
$$(\alpha_i + \lambda_i + \delta) \varphi_i(b,b) = \lambda_i \left[\int_0^b \varphi_i(b-x,b) dF_i(x) + 1 - F_i(x) \right] + \sum_{k=1, k\neq i}^{m} \alpha_{ik} \varphi_k(b,b)$$

Remark 2. When $\delta = 0$ and $w(x,y) = 1$, the expected discounted penalty function denotes the ruin probability $\psi_i(u,b) = P[T_b < \infty | U_b(0) = u]$.Note that for finite b, ruin will occur almost surely, which implies the indicator function can be dropped from the definition of $\varphi_i(u,b)$ and (2) is equivalent to $\psi_i(b,b) = 1$.

Remark 3. Let $w(x,y) = y$, and $\varphi_i(u,b) = E[e^{-\delta T_b} |U_b(T_b)| I(T_b < \infty) | U_b(0) = u]$ can be considered as the discounted expectation of the deficit at ruin.
In $0 \le u < b$

$$(\alpha_i + \lambda_i + \mu_i + \delta)\, \varphi_i(u,b)$$
$$= \lambda_i \left[\int_0^u \varphi_i(u-x,b) dF_i(x) + \int_u^\infty (x-u) dF_i(x) \right]$$
$$+ \mu_i \left[\int_0^{b-u} \varphi_i(u+y,b) dG_i(y) + \varphi_i(b,b) \int_{b-u}^\infty dG_i(y) \right] + \sum_{k=1, k\neq i}^{m} \alpha_{ik} \varphi_k(u,b)$$

At $u = b$

$$(\alpha_i + \lambda_i + \delta) \varphi_i(b,b) = \lambda_i \left[\int_0^b \varphi_i(b-x,b) dF_i(x) + \int_b^\infty (x-b) dF_i(x) \right] + \sum_{k=1, k\neq i}^{m} \alpha_{ik} \varphi_k(b,b)$$

IV. Explicit Result for Exponential Distribution

In this section, we consider the case that the claim amounts and premium are exponentially distributed. We find that the expected discounted penalty function can be explicitly obtained for some specific settings.

Theorem 2. If $F_i(x) = 1 - e^{-\nu_i x}, x \geq 0, \nu_i \geq 0$ and $G_i(y) = 1 - e^{-\beta_i y}, y \geq 0, \beta_i \geq 0$, then the second order differential equation in the matrix form of (1) is given by,

$$(a - \Lambda_1)\Phi''(u,b) + (b - \Lambda_2)\Phi'(u,b) + (c - \Lambda_3)\Phi(u,b) = dW - - - - (3)$$

Where

$$a = diag\{\alpha_1 + \lambda_1 + \mu_1 + \delta, \alpha_2 + \lambda_2 + \mu_2 + \delta, \dots\dots\dots \alpha_m + \lambda_m + \mu_m + \delta\}$$

$$\Lambda_1 = \begin{bmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & 0 & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & 0 \end{bmatrix}; b = (A - \lambda)\nu + (A - \mu)\beta; \lambda = diag\{\lambda_1, \lambda_2, \dots\dots \lambda_m\};$$

$$\mu = diag\{\mu_1, \mu_2, \dots\dots \mu_m\}; \beta = diag\{\beta_1, \beta_2, \dots\dots \beta_m\};$$

$$\nu = diag\{\nu_1, \nu_2, \dots\dots \nu_m\}; \Lambda_2 = \begin{bmatrix} 0 & (\nu_1 - \beta_1)\alpha_{12} & \cdots & (\nu_1 - \beta_1)\alpha_{1m} \\ (\nu_2 - \beta_2)\alpha_{21} & 0 & \cdots & (\nu_2 - \beta_2)\alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ (\nu_m - \beta_m)\alpha_{m1} & (\nu_m - \beta_m)\alpha_{m2} & \cdots & 0 \end{bmatrix};$$

$$c = diag\{-\nu_1\beta_1(\alpha_1 + \delta), -\nu_2\beta_2(\alpha_2 + \delta), \dots\dots\dots -\nu_m\beta_m(\alpha_m + \delta)\};$$

$$\Lambda_3 = \begin{bmatrix} 0 & \nu_1\beta_1\alpha_{12} & \cdots & \nu_1\beta_1\alpha_{1m} \\ \nu_2\beta_2\alpha_{21} & 0 & \cdots & \nu_2\beta_2\alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_m\beta_m\alpha_{m1} & \nu_m\beta_m\alpha_{m2} & \cdots & 0 \end{bmatrix}; d = diag\{\lambda_1\nu_1, \lambda_2\nu_2, \dots\dots \lambda_m\nu_m\};$$

$$W = \begin{bmatrix} w''(u) - \beta_1 w'(u) + \nu_1 w'(u) - \beta_1\nu_1 w(u) \\ w''(u) - \beta_2 w'(u) + \nu_2 w'(u) - \beta_2\nu_2 w(u) \\ \vdots \\ w''(u) - \beta_m w'(u) + \nu_m w'(u) - \beta_m\nu_m w(u) \end{bmatrix}; \Phi''(u,b) = \begin{bmatrix} \varphi_1''(u,b) \\ \varphi_2''(u,b) \\ \vdots \\ \varphi_m''(u,b) \end{bmatrix};$$

$$\Phi'(u,b) = \begin{bmatrix} \varphi_1'(u,b) \\ \varphi_2'(u,b) \\ \vdots \\ \varphi_m'(u,b) \end{bmatrix}; \Phi(u,b) = \begin{bmatrix} \varphi_1(u,b) \\ \varphi_2(u,b) \\ \vdots \\ \varphi_m(u,b) \end{bmatrix}$$

Proof. Considering (1) with exponential claims and exponential premium amount and differentiating it with respect to u, once, twice ,reducing it, gives the following form

$$\begin{aligned} a_i\varphi_i''(u,b) + \varphi_i'(u,b)[(a_i - \lambda_i)\nu_i - (a_i - \mu_i)\beta_i] - \nu_i\beta_i(\alpha_i + \delta)\varphi_i(u,b) \\ = \lambda_i\nu_i[w''(u) - \beta_i w'(u) + \nu_i w'(u) - \beta_i\nu_i w(u)] + \sum_{k=1,k\neq i}^{m} \alpha_{ik}\varphi_k''(u,b) \\ + (\nu_i - \beta_i)\sum_{k=1,k\neq i}^{m} \alpha_{ik}\varphi_k'(u,b) - \nu_i\beta_i \sum_{k=1,k\neq i}^{m} \alpha_{ik}\varphi_k(u,b) \end{aligned}$$

Where $a_i = \alpha_i + \lambda_i + \mu_i + \delta$; $w(u) = \int_u^\infty w(u, x-u)e^{-\nu_i x}dx$; $i = 1,2, \dots .m$

Now rewriting it in the matrix for reduces the above to (3)

Remark 4. If $m = 2$ in (3),then matrix form (3) can be reduced in the following form

For $0 \le u < b$

$$P\Phi''(u,b) + R\Phi'(u,b) + S\Phi(u,b) + T = 0$$

Now let $P = a - \Lambda_1; R = b - \Lambda_2; S = -c + \Lambda_3; T = -dW$

$$P = \begin{bmatrix} \alpha_1 + \lambda_1 + \mu_1 + \delta & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 + \lambda_2 + \mu_2 + \delta \end{bmatrix}$$

$$R = \begin{bmatrix} \alpha_1\nu_1 + \mu_1\nu_1 + \delta\nu_1 - \alpha_1\beta_1 - \lambda_1\beta_1 - \delta\beta_1 & -(\nu_1 - \beta_1)\alpha_{12} \\ -(\nu_2 - \beta_2)\alpha_{21} & \alpha_2\nu_2 + \mu_2\nu_2 + \delta\nu_2 - \alpha_2\beta_2 - \lambda_2\beta_2 - \delta\beta_2 \end{bmatrix}$$

$$S = \begin{bmatrix} -\nu_1\beta_1(\alpha_1 + \delta) & \nu_1\beta_1\alpha_{12} \\ \nu_2\beta_2\alpha_{21} & -\nu_2\beta_2(\alpha_2 + \delta) \end{bmatrix}; T = \begin{bmatrix} -\lambda_1\nu_1(w''(u) - \beta_1 w'(u) + \nu_1 w'(u) - \beta_1\nu_1 w(u)) \\ -\lambda_2\nu_2(w''(u) - \beta_2 w'(u) + \nu_2 w'(u) - \beta_2\nu_2 w(u)) \end{bmatrix}$$

Where

$$\Phi''(u,b) = \begin{bmatrix} {\varphi_1}''(u,b) \\ {\varphi_2}''(u,b) \end{bmatrix}; \Phi'(u,b) = \begin{bmatrix} {\varphi_1}'(u,b) \\ {\varphi_2}'(u,b) \end{bmatrix}; \Phi(u,b) = \begin{bmatrix} \varphi_1(u,b) \\ \varphi_2(u,b) \end{bmatrix}$$

Now again reducing this we get,

$$\Phi'(u) = A\Phi(u) + B ------------------------------ (4)$$

Where $\Phi(u) = \begin{bmatrix} \Phi(u,b) \\ \Phi'(u,b) \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -P^{-1}S & -P^{-1}R \end{bmatrix} ; B = \begin{bmatrix} 0 \\ -P^{-1}T \end{bmatrix}$$

$$-P^{-1}S = \begin{bmatrix} -b_{11} & -b_{12} \\ -b_{21} & -b_{22} \end{bmatrix}; -P^{-1}R = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}; -P^{-1}T = \begin{bmatrix} -c_{11}t_{11} - c_{12}t_{12} \\ -c_{21}t_{11} - c_{22}t_{12} \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence we get the system as

$$\begin{bmatrix} \varphi_1(u,b) \\ \varphi_2(u,b) \\ \varphi_1'(u,b) \\ \varphi_2'(u,b) \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b_{11} & -b_{12} & -a_{11} & -a_{12} \\ -b_{21} & -b_{22} & -a_{21} & -a_{22} \end{bmatrix} \begin{bmatrix} \varphi_1(u,b) \\ \varphi_2(u,b) \\ \varphi_1'(u,b) \\ \varphi_2'(u,b) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -c_{11}t_{11} - c_{12}t_{12} \\ -c_{21}t_{11} - c_{22}t_{12} \end{bmatrix} ----- (5)$$

For $u = b$, the Laplace transform equation for the boundary condition for exponential claim and premium amount distribution satisfies the following equations

$$(\alpha_1 + \lambda_1 + \delta - \frac{\lambda_1 \nu_1}{(s+\nu_1)})\varphi_1{}^*(s) - \varphi_2{}^*(s) = \lambda_1 \left[\frac{w_1^*(s)}{s+\nu_1}\right] - - - - - - - - - (6)$$

$$(\alpha_2 + \lambda_2 + \delta - \frac{\lambda_2 \nu_2}{(s+\nu_2)})\varphi_2{}^*(s) - \varphi_1{}^*(s) = \lambda_2 \left[\frac{w_2^*(s)}{s+\nu_2}\right] - - - - - - - - (7)$$

Where $w_1^*(s)$, $w_2^*(s)$ denotes the Laplace transform of $w(u) = \int_u^\infty w(u, x-u)e^{-\nu_i x}dx$ in two state.

Remark 5. When $\delta > 0$ and $w(x,y) = 1$, the expected discounted penalty function (4) simplifies to Laplace transform of the time to ruin in two state, to be a homogenous first order differential equation of the form $\Phi'(u) = A\Phi(u)$ for $0 \le u < b$ and the boundary condition (6) & (7) at $u = b$,is given by

$$(\alpha_1 + \lambda_1 + \delta - \frac{\lambda_1 \nu_1}{(s+\nu_1)})\varphi_1{}^*(s) - \varphi_2{}^*(s) = \left[\frac{\lambda_1}{s+\nu_1}\right]$$

$$(\alpha_2 + \lambda_2 + \delta - \frac{\lambda_2 \nu_2}{(s+\nu_2)})\varphi_2{}^*(s) - \varphi_1{}^*(s) = \left[\frac{\lambda_2}{s+\nu_2}\right]$$

Remark 6. When $\delta = 0$ and $w(x,y) = 1$, the expected discounted penalty function denotes the ruin probability $\psi_i(u,b) = P[T_b < \infty | U_b(0) = u]$ also reduces (4) to a homogenous first order differential equation as in Remark 5 but the boundary condition is equivalent to $\psi_i(b,b) = 1$ for $i = 1,2$.

Remark 7. Let $w(x,y) = y$, and $\varphi_i(u,b) = E[e^{-\delta T_b}|U_b(T_b)|I(T_b < \infty)|U_b(0) = u]$ can be considered as the discounted expectation of the deficit at ruin also reduces (4) to a homogenous first order differential equation as in Remark 5 but the boundary condition is given as follows

$$(\alpha_1 + \lambda_1 + \delta - \frac{\lambda_1 \nu_1}{(s+\nu_1)})\varphi_1{}^*(s) - \varphi_2{}^*(s) = \left[\frac{\lambda_1}{\nu_1(s+\nu_1)}\right]$$

$$(\alpha_2 + \lambda_2 + \delta - \frac{\lambda_2 \nu_2}{(s+\nu_2)})\varphi_2{}^*(s) - \varphi_1{}^*(s) = \left[\frac{\lambda_2}{\nu_2(s+\nu_2)}\right]$$

V. Numerical illustrations

In this section, we illustrate some results numerically. We consider the two state Markov modulated risk model with stochastic premium and constant barrier for the simplicity of the discussion. Let us give some data analysis about the theoretical results such that we can study the nature of different parameters like Laplace Transform of time to ruin, discounted expectation of the deficit at ruin for the above model. For convenience, we might suppose that $\lambda_1 = 3$, $\lambda_2 = 1$, $\mu_1 = 3$, $\mu_2 = 2$, $\gamma_1 = 3$, $\gamma_2 = 2$, $\beta_1 = 2$, $\beta_2 = 1$, $\alpha_{11} = -1$, $\alpha_{22} = -1$, $\alpha_{12} = 1$, $\alpha_{21} = 1$. We consider the discounted expectation of the deficit at ruin for the model which is the case when $w(x,y) = y$.

Table: 1 gives the data values for barrier $b = 1$ for $\delta = 0, 0.02, 1.5$ and figure 1, 2 ,3 illustrates the detail behaviour of the discounted expectation of the deficit at ruin, φ_1, φ_2 . φ_1 is an increasing function as the initial reserve u increases whereas φ_2 decreases as u increases. We can also conclude that as the interest force δ increases the behaviour of $\varphi_1$and $\varphi_2$is the same and as u increases the discounted expectation of deficit at ruin tends to zero.

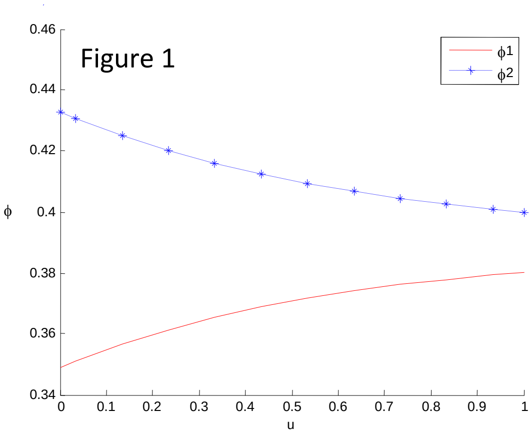
Similarly we have Table 2 and 3 for barrier $b = 5, 10$ and similar conclusions can be drawn for φ_1 and φ_2 so we omit the detailed descriptions.

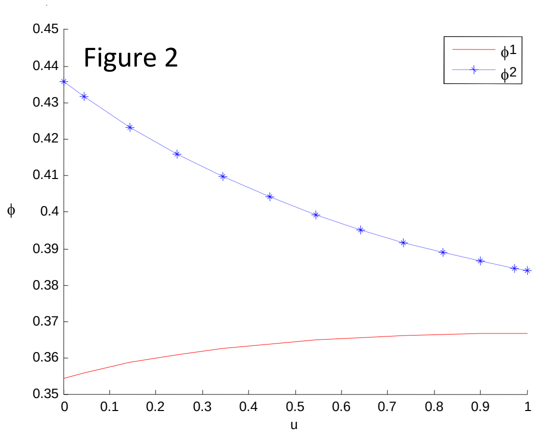
VI. Figures and Tables

Table:1

u	b=1					
	$\delta = 0$		$\delta = 0.02$		$\delta = 1.5$	
	φ_1	φ_2	φ_1	φ_2	φ_1	φ_2
1	0.3801	0.3999	0.3669	0.3840	0.0762	0.0664
0.9	0.3793	0.4009	0.3667	0.3866	0.0826	0.0722
0.8	0.3779	0.4026	0.3665	0.3889	0.0927	0.0813
0.7	0.3762	0.4045	0.3662	0.3918	0.1047	0.0919
0.6	0.3741	0.4068	0.3657	0.3952	0.1187	0.1043
0.5	0.3717	0.4094	0.3650	0.3993	0.1352	0.1186
0.4	0.3688	0.4125	0.3640	0.4041	0.1545	0.1353

0.3	0.3654	0.4160	0.3627	0.4097	0.1770	0.1547
0.2	0.3614	0.4202	0.3610	0.4161	0.2032	0.1771
0.1	0.3566	0.4250	0.3588	0.4234	0.2338	0.2031
0	0.3488	0.4327	0.3546	0.4359	0.2795	0.2419





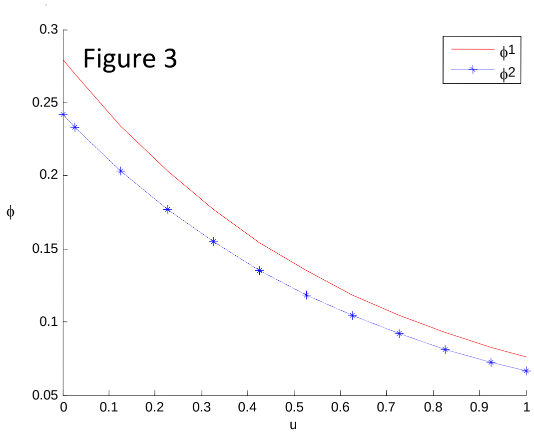
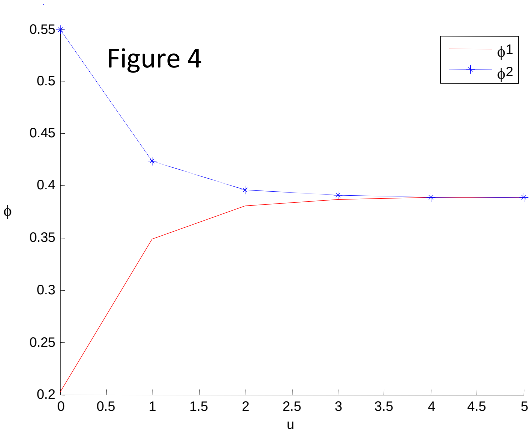
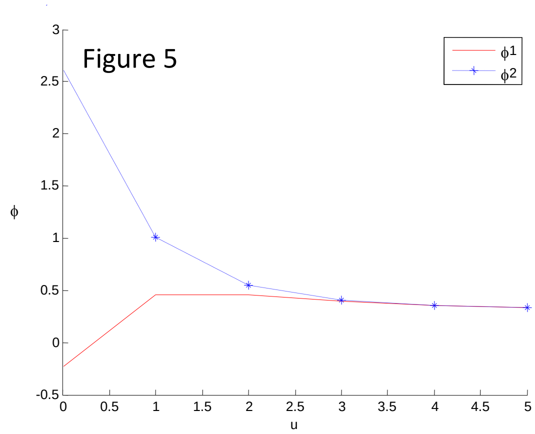


Table :2

u	b=5					
	$\delta = 0$		$\delta = 0.02$		$\delta = 1.5$	
	φ_1	φ_2	φ_1	φ_2	φ_1	φ_2
5	0.3888	0.3890	0.3380	0.3359	0.0010	0.0008
4	0.3885	0.3893	0.3548	0.3533	0.0038	0.0029
3	0.3871	0.3905	0.3940	0.4045	0.0168	0.0119
2	0.3804	0.3962	0.4570	0.5511	0.0764	0.0491
1	0.3492	0.4232	0.4566	1.0061	0.3554	0.2012
0	0.2029	0.5498	-0.2244	2.6042	1.6846	0.8092





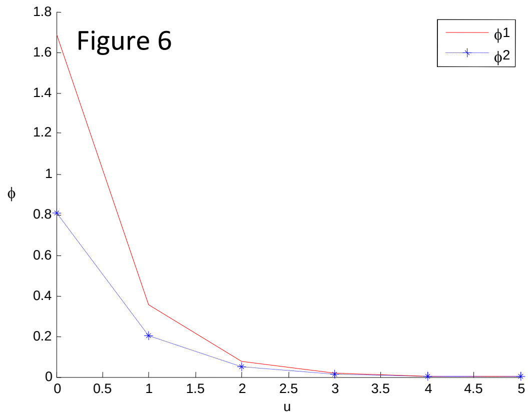
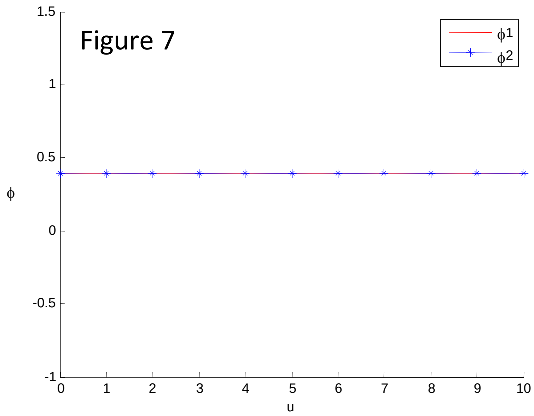
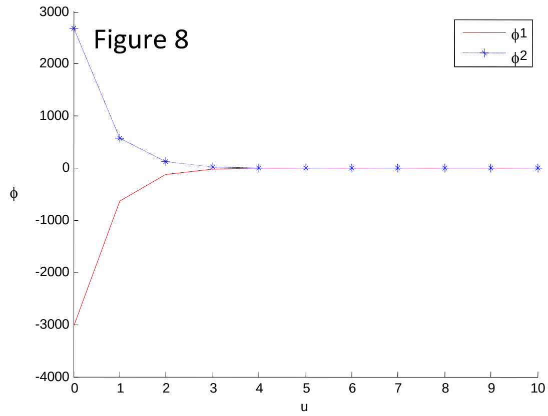
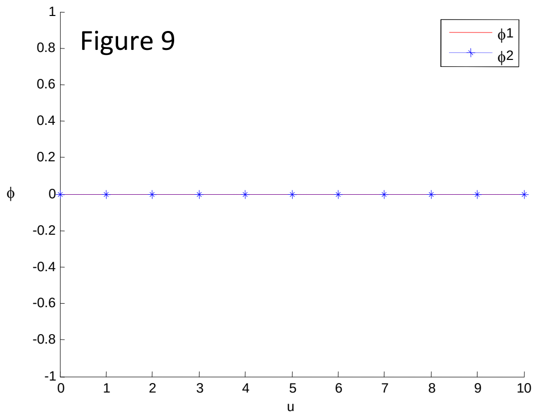


Table:3

u	b=10					
	$\delta = 0$		$\delta = 0.02$		$\delta = 1.5$	
	φ_1	φ_2	φ_1	φ_2	φ_1	φ_2
10	0.3889	0.3889	0.3	0.3	0	0
9	0.3889	0.3889	0.3	0.3	0	0
8	0.3889	0.3889	0.3	0.4	0	0
7	0.3889	0.3889	0.4	0.5	0	0
6	0.3889	0.3889	0.4	0.9	0	0
5	0.3889	0.3889	-0.1	2.2	0	0
4	0.3889	0.3889	-3.9	7.4	0	0
3	0.3889	0.3889	-24.5	29.5	0	0
2	0.3889	0.3889	-128.4	128.4	0	0
1	0.3889	0.3889	-631.7	581.8	0	0
0	0.3889	0.3889	-3030.7	2689.4	0	0







VII. Conclusions

We have supposed a Markov modulated risk model with stochastic premium amount and a constant barrier .We have not only discussed the integral equation of expected discounted penalty function of the model but also offer a data analysis of discounted expectation of deficit at ruin for some special cases. These provide a deep insight into this model, which resembles the state of the present environment, to study different insurance related parameters. As a future scope we can study the other insurance related parameters of this model.

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