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Some approximation results on Otto Szâsz type positive linear operators

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***ABSTRACT** : Recently Deo N.et.al. (Appl. Maths. Compt., 201(2008), 604-612.) introduced a new Bernstein type special operators. Motivated by Deo N.et.al., in this paper we introduce generalization of positive linear operators (1.5) and (1.6) which is the particular case of positive linear operators (1.7) and (1.8). We shall study some approximation results on it.*

I. INTRODUCTION

Recently Deo N.et.al. [1] introduced a new Bernstein type special operators $\{\boldsymbol{V}_n f\}$ defined as,

$$(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

where $$p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}; \qquad for\ 0 \le x \le \frac{n}{n+1}$$

Again Deo N.et.al. [1] gave the integral modification of the operators (1.1) which are defined as ,

$$(L_n f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n.k}(t) f(t) dt \quad \ldots\ldots\ldots\ldots\ldots (1.2)$$

and prove some approximation results on the operators (1.2).

Singh S.P. [4] studied some approximation results on a sequence of Szâsz type operators defined as,

$$(S_{n,x} f)(t) = \sum_{k=0}^{\infty} b_{n,k}(t) f\left(x+\frac{k}{n}\right); \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.3)$$

where $$b_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}; \qquad x \in [0,\infty) is\ fixed.$$

which map the space of bounded continuous funtions $C_B[0,\infty)$ into itself following [3].

Kasana H.S. et. el. [2] obtained a sequence of modified Szâsz operators for integrable function on $[0,\infty)$ defined as,

$$\begin{aligned}(M_{n,x} f)(t) &\equiv M_{n,x}(f(y);t) \\ &= n\sum_{k=0}^{\infty} b_{n,k}(t) \int_0^{\infty} b_{n.k}(y) f(x+y) dy \quad \ldots\ldots\ldots\ldots\ldots (1.4)\end{aligned}$$

$where\ \ t, x \in [0,\infty)\ and\ x\ \ is\ fixed.$

Motivated by Deo N.et.al.[1] we studied a sequence of positive linear operators $\{\boldsymbol{B}_n f\}$ which are defined as,

$$(B_n f)(x) = n\left(1+\frac{1}{n}\right)^2 e^{-(n+p)x} \sum_{k=0}^{\infty} \frac{(n+p)^k x^k}{k!} \int_0^{\frac{n}{n+1}} p_{n.k}(t) f(t) dt \quad \ldots\ldots\ldots\ldots\ldots (1.5)$$

where $$p_{n,k}(t) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} t^k \left(\frac{n}{n+1} - t\right)^{n-k};\ p > 0\ and \qquad for\ \ t \in \left[0, \frac{n}{n+1}\right] \quad .$$

we studied some approximation results on the operators (1.5).

Again following Kasana H.S. et. el. [2] we introduce a sequence of positive linear operators $\{\boldsymbol{B}_{n,x} f\}$ which are defined as,

$$(B_{n,x} f)(t) = n\left(1+\frac{1}{n}\right)^2 e^{-(n+p)t} \sum_{k=0}^{\infty} \frac{(n+p)^k t^k}{k!} \int_0^{\frac{n}{n+1}} p_{n.k}(y) f(x+y) dy \quad \ldots\ldots (1.6)$$

$$where\ \ t\,, x \in \left[0, \frac{n}{n+1}\right]\ and\ x\ \ is\ fixed.$$
and studied some approximation results on the operators (1.6).

If we put $p = 0$ in the operators (1.5) and (1.6), it gives the new modified operators $\{\boldsymbol{B}_n^* f\}\ and\ \{\boldsymbol{B}_{n,x}^* f\}$ as:

$$(B_n^* f)(x) = n\left(1+\frac{1}{n}\right)^2 e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\frac{n}{n+1}} p_{n.k}(t)\, f(t)dt \qquad \dots\dots\dots\dots\dots(1.7)$$

and

$$\left(B_{n,x}^* f\right)(t) = n\left(1+\frac{1}{n}\right)^2 e^{-nt}\sum_{k=0}^{\infty}\frac{(nt)^k}{k!}\int_0^{\frac{n}{n+1}} p_{n.k}(y)\, f(x+y)dy \qquad \dots\dots\dots.\dots(1.8)$$

$$where\ \ t\,, x \in \left[0, \frac{n}{n+1}\right]\ and\ x\ \ is\ fixed.$$

We shall study some approximation results on the operators (1.7) and (1.8).

BASIC RESULTS-I

In order to prove our main result, the following basic results are needed.

1. $e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}k \qquad = nx \qquad \dots\dots\dots\dots\dots(2.1)$
2. $e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}k^2 \qquad = n^2x^2 + nx \qquad \dots\dots\dots\dots\dots(2.2)$
3. $e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}k^3 \qquad = n^3x^3 + 3n^2x^2 + nx \qquad \dots\dots\dots\dots\dots(2.3)$
4. $e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}k^4 \qquad = n^4x^4 + 6n^3x^3 + 7n^2x^2 + nx \qquad \dots\dots\dots..\dots(2.4)$

PROOF OF BASIC RESULTS-I

We know that

$$e^{nx} = \sum_{k=0}^{\infty}\frac{(nx)^k}{k!} \qquad \dots\dots\dots\dots\dots(2.5)$$

Differentiating with respect to x, we get

$$ne^{nx} = \sum_{k=0}^{\infty}\frac{n^k k x^{k-1}}{k!}$$

Multiplying x both sides, we get

$$nxe^{nx} = \sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\ k \qquad \dots\dots\dots\dots\dots\dots(2.6)$$

$$nx = e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\ k$$

This completes the proof of (2.1).

Again differentiating (2.6) with respect to x, we get

$$n^2xe^{nx} + ne^{nx} = \sum_{k=0}^{\infty}\frac{n^k k x^{k-1}}{k!}\ k$$

Multiplying x both sides, we get

$$n^2x^2e^{nx} + nxe^{nx} = \sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\ k^2 \qquad \dots \quad \dots.(2.7)$$

$$[n^2x^2 + nx]e^{nx} = \sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\ k^2$$

$$[n^2x^2 + nx] = e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\ k^2$$

This completes the proof of (2.2).
In the same way after differentiations and calculations, we get required result s (2.3) and (2.4).

BASIC RESULTS-II

1. $(B_n^* 1)(x) = 1 \qquad \dots\dots\dots\dots\dots(2.8)$

2. $(B_n^* t)(x) \to x \quad as \quad n \to \infty.$ ……………(2.9)
3. $(B_n^* t^2)(x) \to x^2 \quad as \quad n \to \infty.$ ……………(2.10)
4. $(B_n^* t^3)(x) = \frac{n^3[n^3x^3+9n^2x^2+18nx+6]}{(n+1)^3 \quad (n+2) \quad (n+3) \quad (n+4)}$ ……………(2.11)
5. $(B_n^* t^4)(x) = \frac{n^4[n^4x^4+16n^3x^3+72\, n^2x^2+96\, nx+24\,]}{(n+1)^4 \quad (n+2) \quad (n+3) \quad (n+4) \quad (n+5)}$ …………..(2.12)
6. $(B_n^*(t-x))(x) = \frac{n[1-3x]-2x}{(n+1) \quad (n+2)}$ ……………(2.13)
7. $(B_n^* \,(t-x)^2)(x) = \frac{n^3(2x-x^2)+n^2[11x^2-8x+2]+n[17x^2-6x]+6x^2}{(n+1)^2 \quad (n+2) \quad (n+3)}$ …. ………(2.14)
8. $(B_n^* \,(t-x)^3)(x) = o\left(\frac{1}{n}\right).$ …………….(2.15)
9. $(B_n^* \,(t-x)^4)(x) = o\left(\frac{1}{n^2}\right).$ …………….(2.16)

Proof of Basic Results-II.

By putting $f(t) = 1$ in equation (1.7), we get

$$(B_n^* 1)(x) = n\left(1+\frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^k \left(\frac{n}{n+1} - t\right)^{n-k} 1dt$$

$$= n\left(1+\frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \; \frac{1}{n} \left(\frac{n}{n+1}\right)^2$$

$$= e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \;\; = 1.$$

This completes the proof of (2.8).

By putting $f(t) = t$ in equation (1.7), we get

$$(B_n^* t)(x) = n\left(1+\frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^k \left(\frac{n}{n+1} - t\right)^{n-k} tdt$$

$$= n\left(1+\frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \; \frac{(k+1)}{n(n+2)} \left(\frac{n}{n+1}\right)^3$$

$$= \frac{n}{(n+1)(n+2)} \left\{ e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \, k + \; e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \, 1 \right\}$$

$$= \frac{n}{(n+1)(n+2)} \; [nx+1]$$

$$= \frac{n^2 x + n}{(n+1)\,(n+2)}$$

$(B_n^* t)(x) \to x \quad as \quad n \to \infty.$

This completes the proof of (2.9).

By putting $f(t) = t^2$ in equation (1.7), we get

$$(B_n t^2)(x) = n\left(1+\frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^k \left(\frac{n}{n+1} - t\right)^{n-k} t^2 \; dt$$

$$= n\left(1+\frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \; \frac{(k+1)(k+2)}{n(n+2)(n+3)} \left(\frac{n}{n+1}\right)^4$$

$$= \frac{n^2}{(n+1)^2(n+2)(n+3)} \left\{ e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \, k^2 + \; e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} 3k + \; e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \; 2 \right\}$$

$$= \frac{n^2[n^2x^2 + 4nx + 2]}{(n+1)^2(n+2)(n+3)}$$

$(B_n^* t^2)(x) \to x^2 \quad as \quad n \to \infty.$

This completes the proof of (2.10).

In the same way by taking $f(t) = t^3$ & $f(t) = t^4$ respectively in (1.7) and after little calculations we get required results (2.11) to (2.16).
This completes the proof.

MAIN RESULTS

In this section we shall give our main results

LEMMA: Let $f \epsilon \mathcal{C}\left[0\ ,\ \frac{n}{n+1}\right]$ then the sequence of positive linear operator defined by $\{B_n^* f\}$ is converges uniformly to f as $n \to \infty$.

Proof : Since from basic results (2.8), (2.9) and (2.10), we get

$$(B_n^* 1)(x) \xrightarrow{uniformly} 1 \quad as \quad n \to \infty$$
$$(B_n^* t)(x) \xrightarrow{uniformly} x \quad as \quad n \to \infty$$
$$(B_n^* t^2)(x) \xrightarrow{uniformly} x^2 \quad as \quad n \to \infty$$

Then using Korokin theorem we can conclude that

$$(B_n^* f)(x) \xrightarrow{uniformly} f \quad as \quad n \to \infty.$$

This completes the proof.

Theorem : Let f be the integrable and bounded in the interval $\left[0\ ,\ \frac{n}{n+1}\right]$ and let if f'' exists at a point x in $\left[0\ ,\ \frac{n}{n+1}\right]$, then one gets that

$$\lim_{n\to\infty} n[(B_n^* f)(x) - f(x)] = (1-3x)f'(x) + \frac{x(2-x)}{2} f''(x).$$

where $\{B_n^* f\}$ are defined in (1.7).

Proof : Since f'' exists at a point x in $\left[0\ ,\ \frac{n}{n+1}\right]$, then by using Taylor's expansion, we write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \lambda(t-x) \qquad \dots\dots\dots\dots(3.1)$$

where $\lambda(t-x) \to 0$ as $t \to x$.

Now for each $\varepsilon > 0$, there corresponds $\delta > 0$ such that

$$|\lambda(t-x)| \le \varepsilon \qquad whernever \quad |t-x| \le \delta.$$

Again for $|t-x| > \delta$, then there exist a positive number M such that

$$|\lambda(t-x)| \le M \le M\frac{(t-x)^2}{\delta^2} \quad .$$

Thus for all t and $x\epsilon\left[0\ ,\ \frac{n}{n+1}\right]$, we get

$$|\lambda(t-x)| \le \varepsilon + M\frac{(t-x)^2}{\delta^2} \qquad \dots\dots\dots\dots\dots(3.2)$$

Applying $\{B_n\}$ on (3.1), we get

$$\begin{aligned}(B_n^* f)(x) &= f(x)(B_n^* 1)(x) + f'(x)\big(B_n^*(t-x)\big)(x) + \frac{f''(x)}{2}(B_n^*(t-x)^2)(x) \\ &\quad + \big(B_n^*(t-x)^2\lambda(t-x)\big)(x) \\ &= f(x) + f'(x)\left[\frac{n[1-3x]-2x}{(n+1)\quad(n+2)}\right] \\ &\quad + \frac{f''(x)}{2}\left[\frac{n^3(2x-x^2)+n^2[11x^2-8x+2]+n[17x^2-6x]+6x^2}{(n+1)^2\quad(n+2)\quad(n+3)}\right] \\ &\quad + n\left(1+\frac{1}{n}\right)^2 e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\int_0^{\frac{n}{n+1}} p_{n.k}(t)(t-x)^2\lambda(t-x)\,dt \quad .\end{aligned}$$

Multiplying n both side, we get

$$n[(B_n^* f)(x) - f(x)] = f'(x)\left[\frac{n[1-3x]-2x}{(n+1)\quad(n+2)}\right]n + \frac{f''(x)}{2}\left[\frac{n^3(2x-x^2)+[11x^2-8x+2]+n[17x^2-6x]+6x^2}{(n+1)^2\quad(n+2)\quad(n+3)}\right]n + nR_n(t,x)$$
$$(say) \qquad \dots\dots\dots(3.3)$$

Here we write,

$$nR_n(t,x) = n\left\{\left(1+\frac{1}{n}\right)^2 e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\int_0^{\frac{n}{n+1}} p_{n.k}(t)(t-x)^2\lambda(t-x)\,dt\right\}$$

$$|nR_n(t,x)| = \left|n\left\{\left(1+\frac{1}{n}\right)^2 e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\int_0^{\frac{n}{n+1}} p_{n.k}(t)(t-x)^2\lambda(t-x)\,dt\right\}\right|$$

$$\leq n\left\{\left(1+\frac{1}{n}\right)^2 e^{-nx}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\int_0^{\frac{n}{n+1}} p_{n.k}(t)|(t-x)^2\lambda(t-x)|\,dt\right\} \quad \ldots(3.4)$$

Using (3.2) in equation (3.4), we get

$$|nR_n(t,x)| \leq n\varepsilon(B_n^*(t-x)^2)(x) + \frac{nM}{\delta^2}(B_n^*(y-t)^4)(x)$$

$$\leq n\varepsilon\, o\left(\frac{1}{n}\right) + \frac{nM}{\delta^2}\, o\left(\frac{1}{n^2}\right)$$

$$\leq \varepsilon + \frac{M}{\delta^2}\, o\left(\frac{1}{n}\right)$$

By choosing $\delta = n^{-1/4}$, we get that

$$|nR_n(t,x)| \leq \varepsilon + \frac{M}{n^{-1/2}}\, o\left(\frac{1}{n}\right)$$

$$\leq \varepsilon + M\, o\left(\frac{1}{\sqrt{n}}\right).$$

Since ε is arbitrary and small, we get

$$|nR_n(t,x)| \to 0 \quad as \quad n \to \infty. \quad \ldots\ldots\ldots\ldots\ldots\ldots..(3.5)$$

Using (3.5) in equation (3.3), we get

$$\lim_{n\to\infty} n[(B_n^* f)(x) - f(x)] = (1-3x)f'(x) + \frac{x(2-x)}{2} f''(x).$$

This completes the proof.

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