IOSR Journal of Mathematics (IOSRJM)
ISSN: 2278-5728 Volume 1, Issue 2 (May-June 2012), PP 16-27
www.iosrjournals.org

A NEW CLASS OF MEROMORPHIC FUNCTIONS USING D^m OPERATOR

Dr. Deepaly Nigam

Abstract: *We introduce a new class* M_p^k *(m, α) of functions of the form* $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+k} z^{n+k}$, $a_{n+k} \geq 0$, *$k \in N_0$ which are regular in the punctured unit disk U* = {Z : Z∈ℂ : 0 < |Z| < 1}. Sharp results concerning coefficients, distortion, closure properties, integral operator, neighborhood property, inclusion property and radii of starlikeness and convexity for the class are determined.*

I. Introduction

Let M_p^k denote the class of functions of the form

(1.1) $$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+k} z^{n+k}; \qquad a_{n+k} \geq 0, p \in N \text{ and } k \in N_0.$$

which are analytic in puntured unit disk U* = {Z : Z∈ℂ : 0 < |Z| < 1}. We define the differential operator D^m on functions belonging to the class M_p^k, using convolution as follows :

(1.2) $$D^m f(z) = \frac{1}{z^p(1-z)^{m+1}} * f(z) \qquad m \in N_0 = \{0,1,2,...\}.$$

or

(1.3) $$D^m f(z) = \frac{1}{z^p m!} \frac{d^m}{dz^m} (z^{m+p} f(z))$$

or

(1.4) $$D^m f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k} z^{n+k}$$

Clearly,
$D^0 f(z) = f(z)$.

A function belonging to the class M_p^k is said to be meromorphically starlike of order 'α' if and only if

(1.5) $$\text{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \quad ; \qquad (z \in D,\ 0 \leq \alpha < p)$$

The subclass of M_p^k consisting of functions which are meromorphically starlike of order α is denoted by $S_p^k(\alpha)$.

We define a class $M_p^k(m, \alpha)$ of function of the form (1.1) which satisfies the condition

(1.6) $$\text{Re}\left\{\frac{D^{m+1} f(z)}{D^m f(z)}\right\} < \frac{m+1+p-\alpha}{m+1}$$

where $m \in N_0$, $p \in N = \{1, 2, \ldots\}$ and $0 \leq \alpha < p$.
Clearly,

$$M_p^k(0, \alpha) \equiv S_p^k(\alpha).$$

Many important properties of certain subclasses of meromorphic p–valent functions were studied by several authors including Aouf and Srivastava [1], Joshi and Srivastava [2], Liu and Srivastava [3], Liu and Owa [4], Liu and Srivastava [5], Owa et.al. [6], and Srivastava et.al. [7].

Extending the work of Liu and Owa [5] we obtain the coefficient inequality, inclusion property, distortion theorems, integral operator, neighbourhood properties, closure properties and radii of starlikeness and convexity for the class $M_p^k(m,\alpha)$.

II. Inclusion Property Of The Class $M_p^k(m,\alpha)$:

For proving the inclusion property, we first prove the following lemma :

Lemma 2.1 : Let $f(z)\in M_p^k$ be given by (1.1), then

(2.1) $$z(D^m f(z))' = (m + 1)D^{m+1}f(z) - (m + p + 1)D^m f(z)$$

Proof : In view of (1.4), we have

$$D^m f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} \quad a_{n+k} z^{n+k}$$

Consider

$$(m+1)D^m f(z) - (m+p+1)f(z) = \frac{(m+1)}{z^p} + \sum_{n=0}^{\infty} \frac{(m+1+p+n+k)!}{(m+1)!(n+p+k)!} a_{n+k} z^{n+k}$$

$$- \frac{(m+p+1)}{z^p} - \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{(m+1)!(n+p+k)!} a_{n+k} z^{n+k}$$

$$= -\frac{p}{z^p} + \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!}(n+k)\, a_{n+k} z^{n+k}$$

$= z(D^m f(z))'$

This completes the proof of lemma.

Theorem 2.2 : $M_p^k(m+1,\alpha) \subset M_p^k(m,\alpha)$ for each $m\in N_0\{0,1,2,\ldots\}$

Proof : Let $f(z)\in M_p^k(m+1,\alpha)$, then

(2.2) $$\operatorname{Re}\left\{\frac{D^{m+2}f(z)}{D^{m+1}f(z)}\right\} < \frac{m+2+p-\alpha}{m+2} \quad , \quad |z|<1$$

Define a regular function w(z) in U = U*U{0} by

(2.3) $$\operatorname{Re}\left\{\frac{D^{m+1}f(z)}{D^m f(z)}\right\} = \frac{(m+1)+(m+1+2p-2\alpha)w(z)}{(1+m)(1+w(z))}$$

Clearly w(0) = 0.

Logarithmic differentiation of (2.3) yields

(2.4) $$\frac{z(D^{m+1}f(z))'}{D^{m+1}f(z)} - \frac{z(D^m f(z))'}{D^m f(z)} = \frac{z(m+1+2p-2\alpha)w'(z)}{(m+1)+(m+1+2p-2\alpha)w(z)} - \frac{zw'(z)}{1+w(z)}$$

Using lemma (2.1), equation (2.4) reduces to

$$(m+2)\frac{D^{m+2}f(z)}{D^{m+1}f(z)} - (m+2+p-\alpha) = \frac{2(p-\alpha)w(z)}{1+w(z)} - (p-\alpha)$$

$$+ \frac{2(p-\alpha)zw'(z)}{\{(m+1)+(m+1+2p-2\alpha)+w(z)\}(1+w(z))}$$

or

$$\frac{\dfrac{D^{m+2}f(z)}{D^{m+1}f(z)} - \dfrac{(m+2+p-\alpha)}{m+2}}{(p-\alpha)/(m+2)} = \frac{2w(z)}{1+w(z)} - 1$$

$$+\frac{2zw'(z)}{\{(m+1)+(m+1+2p-2\alpha)+w(z)\}(1+w(z))}$$

or

(2.5) $$\frac{D^{m+2}f(z)/D^{m+1}f(z)-(m+2+p-\alpha)/m+2}{(p-\alpha)/(m+2)}=\frac{-1+w(z)}{1+w(z)}$$

$$+\frac{2zw^1(z)}{[(m+1)+(m+1+2p-2\alpha)+w(z)](1+w(z))}$$

Suppose there exists a point z_0 in |z| < 1 such that $\max_{|z|<|z_0\}} |w(z)|=|w(z_0)|=1$. From a well known result due to Jack (1971), there is a real number K ≥ 1 such that

(2.6) $z_0w'(z_0) = Kw(z_0)$

From (2.5) and (2.6), we obtain

$$\frac{\dfrac{D^{m+2}f(z_0)}{D^{m+1}f(z_0)}-\dfrac{(m+2+p-\alpha)}{m+2}}{(p-\alpha)/(m+2)}=\frac{-1+w(z_0)}{1+w(z_0)}$$

$$+\frac{2Kw(z_0)}{[(m+1)+(m+1+2p-2\alpha)+w(z_0)](1+w(z_0))}$$

Thus,

$$\operatorname{Re}\left\{\frac{\dfrac{D^{m+2}f(z_0)}{D^{m+1}f(z_0)}-(m+2+p-\alpha)}{(p-\alpha)/(m+2)}\right\}>\frac{1}{2(m+1+p-\alpha)}>0$$

which contradicts (2.2). Hence, |w(z)| < 1 for $z\in U$ and from (2.3), it follows that $f(z)\in M_p^k(m,\alpha)$.

Therefore, $M_p^k(m+1,\alpha)\subset M_p^k(m,\alpha)$

This completes the proof of theorem.

III. Coefficient Inequality

Theorem 3.1 : Let $f(z)\in M_p^k$ be given by (1.1), then $f(z)\in M_p^k$(m, α) if

(3.1) $$\sum_{n=0}^{\infty}\frac{(m+p+n+k)!}{m!(n+p+k)!}\{(n+p+k)+|n+k-p+2\alpha|\}\,a_{n+k}\le 2(p-\alpha)$$

for 0 ≤ α < p.

Proof : For 0 ≤ α < p, consider the expression

(3.2) $$H(z)=|-D^{m+1}f(z)+D^mf(z)|-\left|-D^{m+1}f(z)+\left(\frac{m+1+2p-2\alpha}{m+1}\right)D^mf(z)\right|$$

Replacing $D^m f(z)$ and $D^{m+1}f(z)$ by their series expansions.

$$H(z)=\left|-\sum_{n=0}^{\infty}\frac{(m+p+n+k)!}{m!(n+p+k)!}\frac{(n+p+k)}{(m+1)}a_{n+k}z^{n+k}\right|$$

$$-\left|\frac{2(p-\alpha)}{(m+1)z^p}-\sum_{n=0}^{\infty}\frac{(m+p+n+k)!}{m!(n+p+k)!}\frac{(n+k-p+2\alpha)}{(m+1)}a_{n+k}z^{n+k}\right|$$

for 0 < |z| = r < 1 we have

$$r^p H(z) < \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+k+p)!}\frac{(p+n+k)}{(m+1)} a_{n+k} r^{n+k-p} - \frac{2(p-\alpha)}{(m+1)}$$
$$+\sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+k+p)!}\frac{|n+k-p+2\alpha|}{(m+1)} a_{n+k} r^{n+k-p}$$

Since this holds for all r, $0 < r < 1$, making $r \to 1$ we have

$$H(z) \le \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+k+p)!}\left\{\frac{(p+n+k)|n+k-p+2\alpha|}{(m+1)}\right\} a_{n+k} - \frac{2(p-\alpha)}{(m+1)}$$
$$\le 0 \qquad \text{(using (3.1))}$$

From (3.2), we find

$$\left|\left(-\frac{D^{m+1}f(z)}{D^m f(z)}+1\right)\left(-\frac{D^{m+1}f(z)}{D^m f(z)}+\frac{(m+1+2p-2\alpha)}{(m+1)}\right)^{-1}\right| \le 1$$

or

$$\left|\frac{\left(-\frac{D^{m+1}f(z)}{D^m f(z)}+2\right)-1}{\left(-\frac{D^{m+1}f(z)}{D^m f(z)}+2\right)+\left(1-2\left(\frac{m+1-p+\alpha}{m+1}\right)\right)}\right| \le 1$$

or

$$\mathrm{Re}\left(-\frac{D^{m+1}f(z)}{D^m f(z)}+2\right) > \frac{m+1-p+\alpha}{m+1}$$

or

$$\mathrm{Re}\left(\frac{D^{m+1}f(z)}{D^m f(z)}\right) < \frac{m+1+p-\alpha}{m+1}$$

Hence, $f(z) \in M_p^k(m,\alpha)$.

Theorem 3.2 : Let $f(z) \in M_p^k$ be given by (1.1). Then $f(z) \in M_p^k(m,\alpha)$ if and only if

(3.3) $$\sum_{n=0}^{\infty}(n+k+\alpha)\frac{(m+p+n+k)!}{m!(n+k+p)!} a_{n+k} \le (p-\alpha), \quad a_{n+k} \ge 0$$

for $p/2 \le \alpha < p$.

Proof : In view of theorem (3.1), it is sufficient to show the "only if" part. Let us assume that $f(z) \in M_p^k(m,\alpha)$ then

(3.4) $$\mathrm{Re}\left(\frac{D^{m+1}f(z)}{D^m f(z)}\right) < \frac{m+1+p-\alpha}{m+1} \quad , \quad z \in U.$$

Replacing $D^{m+1}f(z)$ and $D^m f(z)$ by their series expansions, we have

(3.5) $$\mathrm{Re}\left\{\frac{\frac{1}{z^p}+\sum_{n=0}^{\infty}\frac{(m+1+p+n+k)!}{(m+1)!(n+k+p)!}a_{n+k}z^{n+k}}{\frac{1}{z^p}+\sum_{n=0}^{\infty}\frac{(m+p+n+k)!}{(m)!(n+k+p)!}a_{n+k}z^{n+k}}\right\} \le \frac{m+1+p-\alpha}{m+1}$$

when z is real, $\frac{zf'(z)}{f(z)}$ is real and since $a_{n+k} \ge 0$, making $z \to 1^-$ through positive values (3.5) becomes

$$\frac{1+\sum_{n=0}^{\infty}\frac{(m+1+p+n+k)!}{(m+1)!(n+k+p)!}a_{n+k}}{1+\sum_{n=0}^{\infty}\frac{(m+p+n+k)!}{m!(n+k+p)!}a_{n+k}} \le \frac{m+1+p-\alpha}{m+1}$$

hence, we get

$$\sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+k+p)!}(n+k+\alpha)a_{n+k} \le p-\alpha.$$

Hence the result follows.

Remark : The result is sharp. The extremal function being

(3.6) $$f_n(z) = \frac{1}{z^p} + \frac{(p-\alpha)m!(n+k+p)!}{(n+k+\alpha)(m+n+k+p)!} z^{n+k} \quad , \quad , n,k \in N_0 = \{0,1,...\}$$

Corollary 3.3 : If f(z)$\in M_p^k$ with p= 1 and a_k= 0, then theorem 3.2 holds true for $0 \le \alpha < 1$.

IV. DISTORTION THEOREM

Theorem 4.1 : If f(z)$\in M_p^k$(m, α), then

(4.1a) $$\frac{1}{r^p} - \left(\frac{p-\alpha}{k+\alpha}\right) r^k \le |D^m f(z)| \le \frac{1}{r^p} + \left(\frac{p-\alpha}{k+\alpha}\right) r^k \quad , \quad (k \ge 1)$$

and

(4.1b) $$\frac{p}{r^{p+1}} - \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)} r^{k-1} \le |(D^m f(z))'| \le \frac{p}{r^{p+1}} + \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)} r^{k-1}, \quad (k \ge 2)$$

for a < |z| = r < 1 and p/2 ≤ α < p.

Proof : In view of theorem (3.2), we have

(4.2) $$\sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k} \le \frac{p-\alpha}{k+\alpha}$$

Thus, for 0 < |z| = r < 1

$$|D^m f(z)| \le \frac{1}{r^p} + r^k \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k}$$

(4.3) $$|D^m f(z)| \le \frac{1}{r^p} + \frac{(p-\alpha)}{(k+\alpha)} r^k \quad \text{(on using 4.2)}$$

and $$|D^m f(z)| \ge \frac{1}{r^p} - r^k \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k}$$

(4.4) $$\ge \frac{1}{r^p} - \frac{(p-\alpha)}{(k+\alpha)} r^k \quad \text{(on using 4.2)}$$

From (4.3) and (4.4), we get (4.1a).

Again, by theorem (3.2), we have

$$\sum_{n=0}^{\infty} \frac{(n+k)(m+p+n+k)!}{m!(n+p+k)!} a_{n+k} \le (p-\alpha) - \alpha \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k}$$

$$\le (p-\alpha) + \alpha \sum_{n=0}^{\infty} \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k}$$

$$\sum_{n=0}^{\infty} \frac{(n+k)(m+p+n+k)!}{m!(n+p+k)!} a_{n+k} \le (p-\alpha) + \alpha \left(\frac{p-\alpha}{k+\alpha}\right) \quad \text{from (4.2)}$$

$$= \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)}$$

Thus,

(4.5) $$\sum_{n=0}^{\infty} (n+k) \frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k} \le \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)}$$

Further, for using k ≥ 2 and using (4.5), we get and $$|(D^m f(z))'| \le \frac{1}{r^{p+1}} - r^k \sum_{n=0}^{\infty} (n+k)\frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k} r^{n+k-1}$$

(4.6) $$\le \frac{p}{r^{p+1}} + \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)} r^{k-1}$$

and $$|(D^m f(z))'| \ge \frac{p}{r^{p+1}} - r^{k-1} \sum_{n=0}^{\infty} (n+k)\frac{(m+p+n+k)!}{m!(n+p+k)!} a_{n+k}$$

(4.7) $$\ge \frac{p}{r^{p+1}} - \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)} r^{k-1}$$

Using (4.6) and (4.7), we get (4.1b).

Remark : The bounds in (4.1a) and (4.1b) are sharp. Since the equalities are attained for the function

$$f(z) = \frac{1}{z^p} + \frac{(p-\alpha)m!(k+p)!}{(k+\alpha)(m+p+k)!} z^k \qquad (z = \pm r).$$

Corollary 4.2 : If $f(z) \in M_p^k(0,\alpha) \equiv S_p^k(\alpha)$, then

$$\frac{1}{r^p} - \left(\frac{p-\alpha}{k+\alpha}\right) r^k \le |f(z)| \le \frac{1}{r^p} + \left(\frac{p-\alpha}{k+\alpha}\right) r^k \qquad (k \ge 1)$$

and $$\frac{p}{r^{p+1}} - \left(\frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)}\right) r^{k-1} \le |f'(z)| \le \frac{p}{r^{p+1}} + \frac{(p-\alpha)(k+2\alpha)}{(k+\alpha)} r^{k-1} \qquad (k \ge 2).$$

for $0 < |z| = r < 1$ and $p/2 \le \alpha < p$.

For p = 1, we get the following corollary.

Corollary 4.3a : If $f(z) \in M_1^k(m,\alpha)$, then

$$\frac{1}{r} - \left(\frac{1-\alpha}{k+\alpha}\right) r^k \le |D^m f(z)| \le \frac{1}{r} + \left(\frac{1-\alpha}{k+\alpha}\right) r^k \qquad (k \ge 1)$$

and $$\frac{1}{r^2} - \frac{(1-\alpha)(k+2\alpha)}{(k+\alpha)} r^{k-1} \le |D^m f(z)'| \le \frac{1}{r^2} + \frac{(1-\alpha)(k+2\alpha)}{(k+\alpha)} \qquad (k \ge 2).$$

for $0 < |z| = r < 1$ and $1/2 \le \alpha < 1$.

Corollary 4.3b : If $f(z) \in M_1^k(m,\alpha)$ with $a_k = 0$, then corollary 4.3a holds true for $0 < |z| = r < 1$ and $0 \le \alpha < 1$.

for $\alpha = 0$ and m = 0, we get

Corollary 4.4 : If $f(z) \in M_p^k(0,0) \equiv S_p^k(0) \equiv S_p^k$, then

$$\frac{1}{r^p} - \frac{p}{k} r^k \le |f(z)| \le \frac{1}{r^p} + \frac{p}{k} r^k \qquad (k \ge 1)$$

and $$\frac{p}{r^{p+1}} - pr^{k-1} \le |f'(z)| \le \frac{p}{r^{p+1}} + pr^{k-1} \qquad (k \ge 2)$$

for $\alpha = 0$ and p = 1, we get the following corollary.

Corollary 4.5a : If $f(z) \in M_1^k(m,0)$, then

$$\frac{1}{r} - \frac{1}{k} r^{k-1} \le |D^m f(z)| \le \frac{1}{r} + \frac{r^{k-1}}{k} \qquad (k \ge 1)$$

and $$\frac{1}{r} - r^{k-1} \le |D^m f(z)'| \le \frac{1}{r} - r^{k-1} \qquad (k \ge 2)$$

for $0 < |z| = r < 1$ and $½ \le \alpha < 1$.

Corollary 4.5b : If $f(z) \in M_1^k(m,0)$, with $a_k = 0$, then corollary 4.5a holds true for $0 < |z| = r < 1$ and $0 \le \alpha < 1$.

V. CLOSURE THEOREMS :

Theorem 4.1 : If $f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty} a_{n+k}z^{n+k}$, $a_{n+k}\ge 0$

and $$g(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty} b_{n+k}z^{n+k} \quad , \quad b_{n+k}\ge 0$$

are in $M_p^k(m, \alpha)$, then $h(z)=\frac{1}{z^p}+\frac{1}{2}\sum_{n=0}^{\infty}(a_{n+k}+b_{n+k})z^{n+k}$ is also in $M_p^k(m, \alpha)$. For $p/2 \le \alpha < p$.

Proof : f(z) and g(z) being in $M_p^k(m, \alpha)$, we have

(5.1) $$\sum_{n=0}^{\infty}(n+k+\alpha)\frac{(m+n+p+k)!}{m!(n+p+k)!}a_{n+k}\le(p-\alpha)$$

(5.2) and $$\sum_{n=0}^{\infty}(n+k+\alpha)\frac{(m+n+p+k)!}{m!(n+p+k)!}b_{n+k}\le(p-\alpha)$$

for $p/2 \le \alpha < p$.

It is sufficient for h(z) to be a member of $M_p^k(m, \alpha)$ to show

$$\frac{1}{2}\sum_{n=0}^{\infty}(n+k+\alpha)\frac{(m+n+p+k)!}{m!(n+p+k)!}(a_{n+k}+b_{n+k})\le(p-\alpha)$$

which will follow immediately by use of (5.1) and (5.2).

Theorem 5.2 : The class $M_p^k(m, \alpha)$ is closed under convex linear combination.

Proof : Let the function $F_j(z)$ given by

(5.3) $$F_j(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty} a_{n+k,j}z^{n+k} \qquad (j=1,2,...), \quad a_{n+k,j}\ge 0$$

be in $M_p^k(m, \alpha)$. Then it is enough to show that the function

$$H(z)=\lambda F_1(z) + (1-\lambda)F_2(z) \qquad (0 \le \lambda \le 1)$$

is also in $M_p^k(m, \alpha)$. For $0 \le \lambda \le 1$

$$H(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}(\lambda a_{n+k,1}+(1-\lambda)a_{n+k,2})z^{n+k}$$

we observe that

$$\sum_{n=0}^{\infty}\frac{(n+k+\alpha)(m+n+p+k)!}{m!(n+p+k)!}[\lambda a_{n+k,1}+(1-\lambda)a_{n+k,2}]$$

$$=\lambda\sum_{n=0}^{\infty}\frac{(n+k+\alpha)(m+n+p+k)!}{m!(n+p+k)!}\lambda a_{n+k,1}+(1-\lambda)$$

$$\times\sum_{n=0}^{\infty}\frac{(n+k+\alpha)(m+n+p+k)!}{m!(n+p+k)!}a_{n+k,2}$$

$$\le \lambda(p-\alpha) + (1-\lambda)(p-\alpha)$$

$$\le (p-\alpha)$$

By theorem (3.2), we have $H(z)\in M_p^k(m, \alpha)$.

Theorem 5.3 : Let $f_0(z) = \frac{1}{z^p}$ and

$$f_{n+k}(z)=\frac{1}{z^p}+\frac{(p-\alpha)m!(n+k+p)!}{(m+p+n+k)!(n+k+\alpha)}z^{n+k} \quad , \quad n=0,1,2...$$

Then $f(z)\in M_p^k(m, \alpha)$ if and only if f(z) can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \qquad \lambda_1 = \lambda_2 = ... = \lambda_{k-1} = 0$$

where $\sum_{n=0}^{\infty} \lambda_n = 1$, $\lambda_n \geq 0$ and $\lambda_n = 0$ for n = 1, 2, … k–1 (n < k and k ≥ 2).

Proof : Suppose that $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$

thus,

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \lambda_{n+k} \left\{ \frac{(p-\alpha)m!(n+k+p)!}{(m+p+n+k)!(n+k+\alpha)} \right\} z^{n+k}$$

Since

$$\sum_{n=0}^{\infty} \frac{(m+p+n+k)!(n+k+\alpha)}{(p-\alpha)m!(n+k+p)!} \lambda_{n+k} \frac{(p-\alpha)m!(n+k+p)!}{(m+p+n+k)!(n+k+\alpha)} = \sum_{n=0}^{\infty} \lambda_{n+k}$$

$$= \sum_{n=1}^{\infty} \lambda_n$$

$$= 1 - \lambda_0 \leq 1$$

Thus, by theorem (3.2) f(z)∈ $M_p^k(m, \alpha)$.

Conversely : Suppose that the function f(z) defined by (1.1) belongs to $M_p^k(m, \alpha)$, then by theorem (3.2), we have

$$a_{n+k} \leq \frac{(p-\alpha)m!(n+k+p)!}{(m+p+n+k)!(n+k+\alpha)} \quad , \quad n \geq 0.$$

Setting $\lambda_{n+k} = \dfrac{(m+p+n+k)!(n+k+\alpha)a_{n+k}}{(p-\alpha)m!(n+k+p)!}$

with $\lambda_1 = \lambda_2 = ... = \lambda_{k-1} = 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1,\ \lambda_n \geq 0$

then

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+k} z^{n+k}$$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(p-\alpha)m!(n+k+p)!}{(m+p+n+k)!(n+k+\alpha)} \lambda_{n+k} z^{n+k}$$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} \lambda_{n+k} \left\{ f_{n+k} - \frac{1}{z^p} \right\}$$

$$= \frac{1}{z^p} + \left(1 - \sum_{n=0}^{\infty} \lambda_{n+k} \right) + \sum_{n=0}^{\infty} \lambda_{n+k} f_{n+k}$$

$$= \left(1 - \sum_{n=1}^{\infty} \lambda_n \right) \frac{1}{z^p} + \sum_{n=1}^{\infty} \lambda_n f_n$$

$$= \lambda_0 f_0 + \sum_{n=1}^{\infty} \lambda_n f_n$$

$$= \sum_{n=0}^{\infty} \lambda_n f_n(z)$$

This completes the proof.

VI. RADII OF STARLIKENESS AND CONVEXITY :

Theorem 6.1 : Let the function f(z) defi0ned by (1.1) be in the class $M_p^k(m, \alpha)$, then f(z) is meromorphically p–valent starlike of order $\delta(p/2 \le \delta < p)$ in the disk $|z| < r_1$, where

(6.1) $$r_1 = r_1(\alpha, \delta) = \inf_n \left[\frac{(n+k+\alpha)(m+p+n+k)!(p-\delta)}{(n+k+\delta)m!(n+p+k)!(p-\alpha)} \right]^{\frac{1}{n+k+p}} \qquad n \in N_0$$

Proof : It is sufficient to prove that

$$\left| \frac{zf'(z) + pf(z)}{zf'(z) + (2\delta - p)f(z)} \right| \le 1 \qquad \text{for } 0 < |z| < r_1$$

we have

$$\left| \frac{zf'(z) + pf(z)}{zf'(z) + (2\delta - p)f(z)} \right| = \left| \frac{\sum_{n=0}^{\infty} (n+k+p) a_{n+k} z^{n+k}}{\frac{2(\delta - p)}{z^p} + \sum_{n=0}^{\infty} (n+k+2\delta-p) a_{n+k} z^{n+k}} \right|$$

$$= \left| \frac{\sum_{n=0}^{\infty} (n+k+p) a_{n+k} z^{n+k+p}}{-2(p-\delta) + \sum_{n=0}^{\infty} (n+k+2\delta-p) a_{n+k} z^{n+k+p}} \right|$$

Thus, the result follows if

$$\left| \frac{\sum_{n=0}^{\infty} (n+k+p) a_{n+k} z^{n+k+p}}{-2(p-\delta) + \sum_{n=0}^{\infty} (n+k+2\delta-p) a_{n+k} z^{n+k+p}} \right| \le 1$$

or $$\sum_{n=0}^{\infty} (n+k+p) a_{n+k} |z|^{n+k+p} \le 2(p-\delta) - \sum_{n=0}^{\infty} (n+k+2\delta-p) a_{n+k} |z|^{n+k+p}$$

which is equivalent to

$$\sum_{n=0}^{\infty} (n+k+\delta) a_{n+k} |z|^{n+k+p} \le (p-\delta)$$

or

(6.2) $$\sum_{n=0}^{\infty} \frac{(n+k+\delta)}{(p-\delta)} a_{n+k} |z|^{n+k+p} \le 1$$

But, by theorem 3.2, we have

$$\sum_{n=0}^{\infty} \frac{(n+k+\alpha)(m+p+n+k)!}{m!(n+k+p)!(p-\alpha)} a_{n+k} \le 1$$

Hence, (6.2) holds if and only if for all $n \in N_0$

$$\frac{(n+k+\delta)}{(p-\delta)} a_{n+k} |z|^{n+k+p} \le \frac{(n+k+\alpha)(m+p+n+k)!}{m!(n+k+p)!(p-\alpha)} a_{n+k}$$

or $$|z|^{n+k+p} \le \frac{(n+k+\alpha)(m+p+n+k)!(p-\delta)}{(n+k+\delta)m!(n+k+p)!(p-\alpha)}$$

$$|z| \le \left[\frac{(n+k+\alpha)(m+p+n+k)!(p-\delta)}{(n+k+\delta)m!(n+p+k)!(p-\alpha)} \right]^{\frac{1}{n+k+p}}$$

This completes the theorem.

Theorem 6.2 : Let the function f(z) defined by (1.1) be in the class M_p^k (m, α). Then f(z) is meromorphically p–valent convex of order δ(p/2 ≤ δ < p) in the disk |z| < r_2 where $r_2 = r_2(\alpha,\delta)$

$$= \inf_n \left[\frac{(n+k+\alpha)(m+p+n+k)!(p-\delta)p}{(n+k)(n+k+\delta)m!(n+p+k)!(p-\alpha)} \right]^{\frac{1}{n+k+p}} \qquad n \in N_0$$

Theorem 6.3: Let function f(z) given by (1.1) be in the class M_p^k (m, α).

$$F(z) = \frac{1}{c}[(c+p)f(z) + zf^1(z)] \quad , \quad c > 0$$

Then, F(z) is in M_p^k (m, α) for |z| ≤ r(α, β), where

$$r(\alpha,\beta) = \inf_n \left[\frac{(n+k+\alpha)(p-\beta)c}{(n+k+\beta)(c+p+n+k)(p-\alpha)} \right]^{\frac{1}{n+k+p}} \quad , \ n = 0,1,2....$$

The result is sharp for the function $f_n(z)$ given by (3.6).

Proof : $F(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(c+p+n+k)}{c} a_{n+k} z^{n+k}$

Let $$w(z) = 1 - \frac{(m+1)}{(p-\beta)} \left\{ \frac{D^{m+1}F(z)}{D^m F(z)} - 1 \right\}$$

Then it is sufficient to show that

$$\left| \frac{w(z)-1}{w(z)+1} \right| < 1$$

A computation shows that this is satisfied if

$$\sum_{n=0}^{\infty} \frac{(n+k+\beta)(m+p+n+k)!(c+p+n+k)}{(p-\beta)m!(n+k+p)!c} a_{n+k} |z|^{n+p+k} \le 1 \tag{6.3}$$

Since f(z)∈ M_p^k (m, α), by theorem 3.2, we have

$$\sum_{n=0}^{\infty} \frac{(n+k+\alpha)(m+p+n+k)!}{(p-\alpha)m!(n+k+p)!} a_{n+k} \le 1$$

The equation (6.3) is satisfied if

$$\sum_{n=0}^{\infty} \frac{(n+k+\beta)(m+p+n+k)!(c+p+n+k)}{(p-\beta)m!(n+k+p)!c} |z|^{n+p+k} a_{n+k} \le \sum_{n=0}^{\infty} \frac{(n+k+\alpha)(m+p+n+k)!}{(p-\alpha)m!(n+k+p)!} a_{n+k}$$

or

$$|z| \le \left[\frac{(n+k+\alpha)(p-\beta)c}{(n+k+\beta)(c+p+n+k)(p-\alpha)} \right]^{\frac{1}{n+k+p}}$$

VII. INTEGRAL OPERATOR

Theorem 7.1 : Let the function f(z) given by (1.1) be in M_p^k (m, α). Then the integral operator

$$F(z) = (c-p+1)\int_0^1 u^c f(uz)du \qquad \text{for } p \le c < \infty$$

$$0 < u \le 1$$

is in M_p^k (m, δ) where

$$\delta = \frac{p(p+\alpha)(c+k+1) - p(p-\alpha)(c-p+1)}{(c-p+1)(p-\alpha)+(p+\alpha)(c+k+1)}$$

This result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{(p-\alpha)m!(k+p)!}{(k+\alpha)(m+p+k)!} z^{n+k}$$

Proof : Let f(z)∈ M_p^k(m, α). Then

$$F(z) = (c-p+1)\int_0^1 u^c f(uz)du$$

$$= (c-p+1)\int_0^1 \left(\frac{u^{c-p}}{z^p} + \sum_{n=0}^{\infty} a_{n+k} u^{c+n+k} z^{n+k} \right) du$$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{c-p+1}{c+n+k+1} a_{n+k} z^{n+k}$$

It is sufficient to show that

(7.1) $$\sum_{n=0}^{\infty} \frac{(c-p+1)(m+n+p+k)!(n+p+\delta)}{(c+n+k+1)(p-\delta)(n+k+p)!m!} a_{n+k} \le 1$$

Since f(z)∈ M_p^k(m, α), we have

$$\sum_{n=0}^{\infty} \frac{(m+n+p+k)!(n+p+\alpha)}{m!(n+k+p)!(p-\alpha)} \le 1$$

(7.1) is satisfied if

$$(c-p+1)(n+p+\delta)(p-\alpha) \le (c+n+k+1)(p-\delta)(n+p+\alpha)$$

or

$$\delta \le \frac{p(c+n+k+1)(n+p+\alpha) - (c-p+1)(n+p)(p-\alpha)}{(c-p+1)(p-\alpha)+(c+n+k+1)(n+p+\alpha)} = F(n)$$

A computation shows that

F(n + 1) – F(n)

$$= \frac{(p-\alpha)(c-p+1)[(n+p)(n+2p+1)+p(\alpha+p+1)+pn+k(p-\alpha)]}{[(c-p+1)(p-\alpha)+(c+n+k+2)(n+1+p+\alpha)][(c-p+1)(p-\alpha)+(c+n+k+1)(n+p+\alpha)]} > 0$$

for all n ≥ 0. This means F(n) is increasing and F(n) ≥ F(0). Using this the result follows.

VIII. NEIGHBOURHOODS FOR THE CLASS $M_p^k(m, \alpha, \gamma)$:

Definition 8.1 : A function f(z)∈ M_p^k is said to be in the class M_p^k(m, α, γ) if there exist a function g(z)∈ M_p^k(m, α) such that

(8.1) $$\left| \frac{f(z)}{g(z)} - 1 \right| \le 1-\gamma \quad , \quad 0 \le \gamma < 1.$$

Definition 8.2 : $N_\delta(f)$ denote the δ–neighbourhood of the function f∈ M_p^k of the form (1.1), i.e.

(8.2) $$N_\delta(f) = \left\{ g \in M_p^k : g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} b_{n+k} z^{n+k} \right.$$

$$\left. \text{and} \quad \sum_{n=0}^{\infty} (n+k) | a_{n+k} - b_{n+k} | \le \delta \right\}$$

Theorem 8.1 : If g(z)∈ M_p^k(m, α) and

(8.3) $$\gamma = 1 - \frac{\delta(k+\alpha)(m+k+p)!}{k[(k+\alpha)(m+k+p)! - m!(p-\alpha)(k+p)!]}$$

Then $N_\delta(g) \subset M_p^k(m, \alpha, \gamma)$.

Proof : Let $f(z) \in N_\delta(g)$, then we find from (8.2) that

(8.4) $$\sum_{n=0}^{\infty} |a_{n+k} - b_{n+k}| \leq \frac{\delta}{k}$$

Since $g(z) \in M_p^k(m, \alpha)$, we have from (3.3)

(8.5) $$\sum_{n=0}^{\infty} b_{n+k} \leq \frac{(p-\alpha)m!(k+p)!}{(k+\alpha)(m+p+k)!}$$

So that $$\left|\frac{f(z)}{g(z)} - 1\right| = \left|\frac{\sum_{n=0}^{\infty}(a_{n+k} - b_{n+k})}{1 + \sum_{n=0}^{\infty} b_{n+k}}\right|$$

$$\leq \frac{\sum_{n=0}^{\infty}|a_{n+k} - b_{n+k}|}{1 - \sum_{n=0}^{\infty} b_{n+k}}$$

$$= \frac{\delta}{k\left[1 - \frac{(p-\alpha)m!(k+p)!}{(k+\alpha)(m+k+p)!}\right]} \quad \text{(using (8.4) \& (8.5))}$$

$$= \frac{\delta(k+\alpha)(m+k+p)!}{k[(k-\alpha)(m+k+p)! - (p-\alpha)m!(k+p)!]}$$

$= 1 - \gamma$

where γ is given by (8.3). Hence by definition 8.1 $f(z) \in M_p^k(m, \alpha, \gamma)$ which completes the proof.

References

[1]. M.K. AOUF and H.M. SRIVASTAVA, A new criterion for meromorphically p–valent convex functions of order α, Math.Sci.Res. Hot. Line 1(8)(1997), 7–12.

[2]. S.B. JOSHI and H.M. SRIVASTAVA, A certain family of meromorphically multivalent functions, Computers Math. App. 38(3/4) (1999), 201–211.

[3]. J.L. LIU and H.M. SRIVASTAVA, A linear operator and associated families of meromorphically multivalent functions. J. Math. Anal. Appl. 259(2001), 566–581.

[4]. J.L. LIU and S. OWA, On a class of meromorphic p–valent functions involving certain linear operators. Intermat. J. Math. Math. Sci. 32(2002), 271–280.

[5]. J.L. LIU and H.M. SRIVASTAVA, Some convolution conditions for starlikeness and convexity of meromorphically multivalent functions, Applied Math. Letters, 16(2003), 13–16.

[6]. S. OWA, H.E. DARWISH and M.K. AOUF, Meromorphic multivalent functions with positive and fixed second coefficients, Math. Japan, 46(1997), 231–236.

[7]. H.M. SRIVASTAVA, H.M. HOSSEN and M.K. AOUF, A unified presentation of some classes of meromorphically multivalent functions, Computers Math. Appl. 38(11/12)(1999), 63–70.