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On A Subclass of Meromorphic Starlike Univalent Functions With Alternating Coefficients

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Abstract: *Coefficient inequalities and distortion theorems are obtained for certain subclass of meromorphic starlike univalent functions with alternating coefficients. Further class preserving integral operators are obtained.*
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I. Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \qquad \textbf{(1.1)}$$

which are regular in *E= { z: 0<| z | < 1}*.
Define

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = \frac{1}{z} + 3a_1 z + 4a_2 z^2 + ... = \frac{(z^2 f(z))'}{z}$$

$$D^2 f(z) = D(D' f(z))$$

and for n=1,2,3,…..

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m = \frac{(z^2 D^{n-1} f(z))'}{z}.$$

In [9] Uralegaddi and Somanatha obtained a new criteria for meromorphic starlike univalent functions via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha)$, $(0 \le \alpha < 1)$, $n \in N_0 = \{0,1,2,...\}$, where $B_n(\alpha)$ is the class consisting of functions in $\sum$ satisfying

$$\operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 2 \right\} < -\alpha, \ |z| < 1, \ (0 \le \alpha < 1), \ n \in N_0 = \{0,1,2,...\}. \qquad (1.2)$$

We note that $B_0(\alpha) = \sum^*(\alpha)$, is the class of meromorphically starlike functions of order α ($0 \le \alpha < 1$),

and $B_0(0) = \sum^*$ is the class of meromorphically starlike functions. Let σ_A be the subclass of $\sum$ which consists of functions of the form

$$f(z) = \frac{1}{z} + a_1 z - a_2 z^2 + a_3 z^3 - ..... = \frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} a_m z^m \quad , a_m \ge 0 \qquad (1.3)$$

Further let $\sigma^*_{A,n}(\alpha,\beta) = B_n(\alpha,\beta) \cap \sigma_A$.

Definition1: Let $f(z)$ be defined by (1.3).Then $f(z) \in \sigma^*_{A,n}(\alpha,\beta)$ if and only

$$\left| \frac{\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1} f(z)}{D^n f(z)} + 2\alpha - 3} \right| < \beta \quad \text{for } |z| < 1, 0 \le \alpha < 1, \ 0 < \beta \le 1.$$

In the present paper coefficient inequalities ,distortion theorem and closure theorems for the class $\sigma^*_{A,n}(\alpha,\beta)$ are obtained.Techniques used are similar to those Silverman [8].Finally, the class preserving integral operators of the form

$$F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt \quad (c>0). \tag{1.4}$$

II. Coefficient Inequalities

Theorem 1.Let $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$ .If

$$\sum_{m=1}^{\infty}(m+2)^n\left[(1+\beta)m+(2\alpha-1)\beta+1\right]\left|a_m\right|\le 2\beta(1-\alpha) \tag{2.1}$$

Then $f(z)\in B_n(\alpha,\beta)$.

Proof: Suppose that equation (2.1) holds for all admissable values of α,β and n. Consider the expression

$$H(f,f')=\left|D^{n+1}f(z)-D^n f(z)\right|-\beta\left|D^{n+1}f(z)+(2\alpha-3)D^n f(z)\right| \tag{2.2}$$

Replacing $D^n f(z)$ and $D^{n+1}f(z)$ by their series expansions, we have, for $0<|z|=r<1$,

$$H(f,f')=\left|\sum_{m=1}^{\infty}(m+2)^n(m+1)a_m z^{m+1}\right|-\beta\left|2(1-\alpha)-\sum_{m=1}^{\infty}(m+2)^n(m-1+2\alpha)a_m z^{m+1}\right|$$

$$\le\sum_{m=1}^{\infty}(m+2)^n(m+1)\left|a_m\right|r^{m+1}-\beta\{2(1-\alpha)-\sum_{m=1}^{\infty}(m+2)^n(m-1+2\alpha)\left|a_m\right|r^{m+1}\}$$

.Since the above inequality holds for all r (0 < r <1), letting $r\to 1$, we have

$$H(f,f')\le\sum_{m=1}^{\infty}(m+2)^n[(1+\beta)m+(2\alpha-1)\beta+1]\left|a_m\right|-2\beta(1-\alpha)$$

≤ 0 , by (2.1).

Hence it follows that

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right|<\left|\frac{D^{n+1}f(z)}{D^n f(z)}+(2\alpha-3)\right|,$$

which shows that $f(z)\in B_n(\alpha,\beta)$ .Hence the Theorem is completely proved. For functions in $\sigma^*_{A,n}(\alpha,\beta)$ the converse of the above Theorem is also true.

Theorem 2. A function $f(z)$ in σ_A is in $\sigma^*_{A,n}(\alpha,\beta)$ if and only if

$$\sum_{m=1}^{\infty}(m+2)^n\left[(1+\beta)m+(2\alpha-1)\beta+1\right]a_m\le 2\beta(1-\alpha). \tag{2.3}$$

Proof: In view of Theorem 1, it is sufficient to prove that only if part. Let us assume that $f(z)$ is in $\sigma^*_{A,n}(\alpha,\beta)$. Then

$$\left|\frac{\dfrac{D^{n+1}f(z)}{D^n f(z)}-1}{\dfrac{D^{n+1}f(z)}{D^n f(z)}+2\alpha-3}\right|=\left|\frac{\sum_{m=1}^{\infty}(-1)^{m-1}(m+2)^n(m+1)a_m z^{m+1}}{2(1-\alpha)-\sum_{m=1}^{\infty}(-1)^{m-1}(m+2)^n(m-1+2\alpha)a_m z^{m+1}}\right|<\beta.$$

Using the fact that $\operatorname{Re}(z)\le|z|$ for all z,it follows that

$$\operatorname{Re}\left\{\frac{\sum_{m=1}^{\infty}(-1)^{m-1}(m+2)^n(m+1)a_m z^{m+1}}{2(1-\alpha)-\sum_{m=1}^{\infty}(-1)^{m-1}(m+2)^n(m-1+2\alpha)a_m z^{m+1}}\right\}<\beta, \quad z\in E. \qquad (2.4)$$

Now choose the values of z on the real axis so that $\left(\frac{D^{n+1}f(z)}{D^n f(z)}-2\right)$ is real.Upon clearing the denominator in (2.4) and letting $z\to -1$ through real values , we obtain

$$\sum_{m=1}^{\infty}(m+2)^n(m+1)a_m \le \beta[2(1-\alpha)-\sum_{m=1}^{\infty}(m+2)^n(m-1+2\alpha)a_m]$$

$$\sum_{m=1}^{\infty}(m+2)^n\left\{(1+\beta)m+(2\alpha-1)\beta+1\right\}a_m \le 2\beta(1-\alpha).$$

This completes the proof of the Theorem.

Corollary 1.Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,n}(\alpha,\beta)$ .Then

$$a_m \le \frac{2\beta(1-\alpha)}{(m+2)^n\left[(1+\beta)m+(2\alpha-1)+1\right]}\quad , \text{m=1,2,3,…..} \qquad (2.5)$$

Equality holds for the function of the form

$$f_m(z)=\frac{1}{z}+(-1)^{m-1}\frac{2\beta(1-\alpha)}{(m+2)^n\left[(1+\beta)m+(2\alpha-1)+1\right]}z^m\ . \qquad (2.6)$$

III. Distortion Properties and Radius of Convexity

Theorem 3. Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,n}(\alpha,\beta)$ .Then for $0<|z|=r<1,$

$$\frac{1}{r}-\frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}r \le |f(z)| \le \frac{1}{r}+\frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}r \qquad (3.1)$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}z\ , \qquad \text{at } z = r, ir\ . \qquad (3.2)$$

Proof: Suppose $f(z)$ is in $\sigma^*_{A,n}(\alpha,\beta)$. In view of Theorem 2, we have

$$\sum_{m=1}^{\infty}a_m \le \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}, \qquad (3.3)$$

Then for $0<|z|=r<1,$

$$|f(z)|=\left|\frac{1}{z}+\sum_{m=1}^{\infty}(-1)^{m-1}a_m z^m\right|$$

$$\le\left|\frac{1}{z}\right|+\sum_{m=1}^{\infty}a_m|z|^m \le \frac{1}{r}+r\sum_{m=1}^{\infty}a_m$$

$$\le \frac{1}{r}+\frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}r$$

This gives the right hand side inequality of (3.1).

Also,

$$\left| f(z) \right| = \left| \frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} a_m z^m \right|$$
$$\geq \left| \frac{1}{z} \right| - \sum_{m=1}^{\infty} a_m \left| z \right|^m$$
$$\geq \frac{1}{r} - r \sum_{m=1}^{\infty} a_m$$
$$\geq \frac{1}{r} - \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)} r$$

which gives the left hand side inequality of (3.1) .This completes the proof. Putting $n = 0$ and $\beta = 1$ in the above Theorem, we have the following.

Corollary 2.Let the function $f(z)$ defined by (1.3) be in the class $\sigma^{*}_{A,0}(\alpha,1) = \sigma^{*}_{A}(\alpha)$ .Then for $0 < |z| = r < 1,$

$$\frac{1}{r} - \frac{(1-\alpha)}{(1+\alpha)} r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\alpha)}{(1+\alpha)} r$$

The result is sharp.

We observe that our result in corollary 2 improves the result of Uralegaddi and Ganigi[10]

Theorem 4. Let the function $f(z)$ defined by (1.3) be in the class $\sigma^{*}_{A,n}(\alpha,\beta)$ .Then for $0 < |z| = r < 1,$

$$\frac{1}{r^2} - \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)} r \leq |f'(z)| \leq \frac{1}{r^2} + \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)} r .$$

The result is sharp,the extremal function being of the form (3.2).

Proof: From Theorem 2, we have

$$3^n(1+\alpha\beta)\sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} (m+2)^n \left\{(1+\beta)m + (2\alpha-1)\beta + 1\right\} a_m \leq \beta(1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} m a_m \leq \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}$$

Consequently,we obtain

$$|f'(z)| \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1} \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m$$
$$\leq \frac{1}{r^2} + \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)} .$$

Also,

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1} \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m$$

$$\geq \frac{1}{r^2} - \frac{\beta(1-\alpha)}{3^n(1+\alpha\beta)}$$

This completes the proof.

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