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Some Domination Parameters of Arithmetic Graph Vn

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Abstract: *Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern. Nathanson [1] was the pioneer in introducing the concepts of Number Theory, particularly, the 'Theory of Congruences' in Graph Theory, thus paving way for the emergence of a new class of graphs, namely "Arithmetic Graphs". Inspired by the interplay between Number Theory and Graph Theory several researchers in recent times are carrying out extensive studies on various Arithmetic graphs in which adjacency between vertices is defined through various arithmetic functions.*
Keywords*: Arithmetic graph, Domination, Total domination, Independent domination, Connected domination.*

I. Arithmetic V_n graph and its properties

Vasumathi [2] introduced the concept of Arithmetic V_n graphs and studied some of its properties. Their definition of Arithmetic V_n graph is as follows.

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots.. p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider Arithmetic V_n graph without vertex 1 as the contribution of this isolated vertex is nothing when domination parameters are studied.

Clearly, V_n graph is a connected graph. If n is a prime, then V_n graph consists of a single vertex. Hence it is connected. In other cases, by the definition of adjacency in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n.

In this paper we discuss various domination parameters of Arithmetic V_n graph. While studying several properties of Arithmetic V_n graph, it is observed that the domination parameters like domination number, total domination number, independent domination number and connected domination number of these graphs are functions of k, where k is the core of n, that is the number of distinct prime divisors of n.

Let $G(V_n)$ denote the V_n graph throughout this paper.

II. Domination in Arithmetic V_n graph

The concept of domination in graph theory was formalized by Berge [3] and Ore [4] and is strengthened by Haynes, Hedetniemi, Slater [5, 6] who presented a survey articles in the wide field of domination in graphs. Domination in graphs has been studied extensively and at present it is an emerging area of research in graph theory.

Let G(V,E) be a graph. A subset D of V is said to be a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

In this section we find minimum dominating sets of $G(V_n)$ graph and obtain their domination numbers in various cases.

Theorem 1: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots.. p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are primes and $\alpha_1, \alpha_2, \dots. \alpha_k$ are integers ≥ 1, then the domination number of $G(V_n)$ is given by

$$\gamma(G(V_n)) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i} \\ k & \text{Otherwise.} \end{cases}$$

where k is the core of n.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots.. p_k^{\alpha_k}$. Then we have the following possibilities.

Case 1: Suppose $\alpha_i > 1$ for all i. Then we show that the set $D = \{ p_1, p_2, \dots\dots, p_k \}$ becomes a dominating set of $G(V_n)$.

By the definition of $G(V_n)$ graph, it is obvious that the vertices in $G(V_n)$ are primes $p_1, p_2, \dots\dots, p_k$, their powers and their products.

Let V be the vertex set of $G(V_n)$. All the vertices $u \in \langle V - D \rangle$, for which GCD $(u, p_1) = p_1$ are adjacent to the vertex p_1 in D. All the vertices $v \in \langle V - D \rangle$, for which GCD $(v, p_2) = p_2$ are adjacent to the vertex p_2 in D. Continuing in this way we obtain that all the vertices $w \in \langle V - D \rangle$, for which GCD $(w, p_k) = p_k$ are adjacent to the vertex p_k in D. Since every vertex in $\langle V - D \rangle$ has atleast one prime factor viz., $p_1, p_2, \ldots, p_k$ (as they are divisors of n) every vertex in $V - D$ is adjacent to at least one vertex in D. Thus D becomes a dominating set of $G(V_n)$.

We now prove that D is minimum. Suppose we remove any p_i from D. Then the vertices of the form p_i^r, $r > 1$ will be non-adjacent to any other vertex p_j as GCD $(p_i^r, p_j) = 1$ for $i \neq j$. Therefore every $p_i, i = 1, 2, \ldots, k$ must be included into D. If we form a minimum dominating set in any other manner, then the order of such a set is not smaller than that of D. This follows from the properties of prime divisors of a number. Hence $\gamma(G(V_n)) = |D| = k$.

Case 2: Suppose $\alpha_i = 1$ for only one i. That is, p_i is the only prime divisor of n with exponent 1. Then $n = p_1^{\alpha_1} . p_2^{\alpha_2} \ldots . p_{i-1}^{\alpha_{i-1}} . p_i . p_{i+1}^{\alpha_{i+1}} . \ldots . p_k^{\alpha_k}$.

Now the primes $p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k$ are to be included into a dominating set D of $G(V_n)$ as proved in Case 1. But the vertex p_i will not be adjacent to any other vertex p_j in D. Therefore p_i is also to be included into D. Thus the set $\{p_1, p_2, \ldots, p_k\}$ forms a minimum dominating set of $G(V_n)$.

Hence $\gamma(G(V_n)) = k$.

Case 3: Suppose $\alpha_i = 1$ for more than one i. Denote the prime divisors of n with exponent 1 by $p_1, p_2, \ldots, p_i$ and write $n = p_1 . p_2 . \ldots p_i . p_{i+1}^{\alpha_{i+1}} . \ldots . p_k^{\alpha_k}$.

Let $D = \{p_1, p_2, \ldots, p_{i-2}, p_{i-1}, p_i, p_{i+1} \ldots, p_k\}$. Then we show that D forms a minimum dominating set of $G(V_n)$. Any vertex in $V - D$ will be of the form $p_1^{a_1} . p_2^{a_2} \ldots . p_i^{a_i} . p_{i+1}^{a_{i+1}} . \ldots . p_k^{a_k}$ where $a_1, a_2, \ldots, a_i \leq 1$, and $a_j \leq \alpha_j$ for $j = i + 1, \quad i + 2, \ldots, k$. Then clearly D is a dominating set as every vertex in $V - D$ is adjacent to at least one vertex in D. However this is not a minimum dominating set.

Let $D' = \{p_1, p_2, \ldots, p_{i-2}, p_{i-1} . p_i, p_{i+1}, \ldots, p_k\}$ where the vertices p_{i-1}, p_i are adjacent to the vertex $p_{i-1} . p_i$. This is clearly a minimum dominating set of $G(V_n)$. For, any deletion of vertices in this set will not make it a dominating set any more.

Hence $\gamma(G(V_n)) = |D'| = k - 1$. ∎

III. Total Domination in Arithmetic V_n graph

Total dominating sets are introduced by Cockayane, Dawes and Hedetniemi [7]. Some results regarding total domination can be seen in [8].

Let $G(V, E)$ be a graph without isolated vertices. Then a total dominating set T is a subset of V such that every vertex of V is adjacent to some vertex in T. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

In this section we discuss about the total domination of $G(V_n)$ and obtain its total domination number.

Theorem 2 : Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots p_k^{\alpha_k}$ where $\alpha_i \geq 1, \forall i$. Then the total domination number of $G(V_n)$ is k, where k is the core of n.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots p_k^{\alpha_k}$ where $\alpha_i \geq 1, \quad \forall i$. Let $T = \{p_1, p_2, \ldots, p_{k-1}, t\}$ where $t = p_1 \, p_2 \ldots . p_k$, the product of k distinct prime divisors of n , which is a vertex in $G(V_n)$.

We now show that T is a minimum total dominating set of $G(V_n)$. As in the proof of Case 1 of Theorem 1, we see that a vertex $u \in V$ for which GCD $(u, p_i) = p_i$ is adjacent to the vertex p_i in D for $i = 1, 2, \ldots, k - 1$. Now the vertex p_k and the vertices which are powers of p_k are adjacent to the vertex $t = p_1 \, p_2 \ldots . p_k$, as GCD $(t, p_k^r) = p_k$ for $r = 1, 2, \ldots, \alpha_k$. Also the vertices $p_1, p_2, \ldots, p_{k-1}$ are adjacent to vertex t respectively. Thus T forms a total dominating set of $G(V_n)$.

We next prove that T is minimum. Suppose we remove any vertex p_i from T. Then the vertices of the form $p_i \, p_j$ are not adjacent to any vertex in T. That is T is not a dominating set, a contradiction. Further if we form a minimum total dominating set in any other manner, by the properties of prime numbers, the order of such a set is not smaller than that of T. Thus the total domination number of $G(V_n)$ is k.

That is $\gamma_t(G(V_n)) = |T| = k$. ∎

IV. Independent Domination in Arithmetic V_n graph

Independent domination was introduced by R.B.Allan and R.C.Laskar [9]. We determine minimum independent dominating sets and independent domination number of $G(V_n)$ graph as follows.

A dominating set D of a graph G in which no two vertices are adjacent is called an independent dominating set of G. The induced subgraph $\langle D \rangle$ is a null graph if D is an independent dominating set.

The minimum cardinality of an independent dominating set of G is called the independent domination number of G and is denoted by $\gamma_i(G)$.

Theorem 3: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots. p_k^{\alpha_k}$, where $p_1, p_2, \ldots p_k$ are primes and $\alpha_1, \alpha_2, \ldots. \alpha_k$ are integers ≥ 1, then the independent domination number of $G(V_n)$ is given by

$$\gamma_i\left(G(V_n)\right) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i} \\ k & \text{Otherwise} \end{cases}$$

where k is the core of n.

Proof: Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots. p_k^{\alpha_k}$. We have the following cases.

Case 1: Suppose $\alpha_i > 1$, $\forall$ i or $\alpha_i = 1$ for only one i.

Let $D_i = \{ p_1, p_2, \ldots\ldots, p_k\}$. Then as proved in Case 1 and Case 2 of Theorem 1, D_i with cardinality k becomes a minimum dominating set of $G(V_n)$.

Now any two vertices p_i, p_j in D_i for $i \neq j$ are not adjacent to each other because GCD $\left(p_i, p_j\right) = 1$ for $i \neq j$. Hence D_i becomes an independent dominating set of $G(V_n)$ with minimum cardinality.

Hence $\gamma_i\left(G(V_n)\right) == |D_i| = k$.

Case 2: Suppose $\alpha_i = 1$ for more than one i. Let $p_1, p_2, \ldots\ldots, p_i$ are the prime divisors of n with exponent 1 and writing these i primes in ascending order, we have $n = p_1. p_2. \ldots\ldots p_i. p_{i+1}^{\alpha_{i+1}}. \ldots\ldots p_k^{\alpha_k}$. Then recall a dominating set D_i with minimum cardinality $k-1$ defined as in case 3 of Theorem 1.

Let $D_i = \{ p_1, p_2, \ldots\ldots, p_{i-2}, p_{i-1}. p_i, p_{i+1} \ldots\ldots, p_k \}$. As in case 1, no two vertices in D_i are adjacent to each other because for $i \neq j$, GCD $\left(p_i, p_j \right) = 1$. Therefore D_i becomes an independent dominating set of $G(V_n)$, with minimum cardinality.

Hence $\gamma_i\left(G(V_n)\right) == |D_i| = k-1$. ■

V. Connected Domination in Arithmetic V_n graph

Sampathkumar, E. and Walikar,H.B. [10] introduced the concept of connected domination in graphs. For a survey of connected domination see [11]. It is easy to observe that only connected grapgs have a connected dominating set.

A dominating set D of a graph G is said to be a connected dominating set if the induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

We now study connected dominating sets of $G(V_n)$ graph and obtain connected domination number in various cases as follows.

Theorem 4: If n is power of a prime, then the connected domination number of $G(V_n)$ is 1.

Proof: Suppose $n = p^\alpha$, where p is any prime. Let V be the vertex set of $G(V_n)$ graph given by V = $\{ p, p^2, p^3 \ldots\ldots, p^\alpha \}$. Now GCD $(p, p^j) = p$, for all $j > 1$, implies that vertex p is adjacent to the remaining vertices $p^2, p^3 \ldots\ldots, p^\alpha$. Hence if $D_c = \{ p \}$ then D_c becomes a dominating set of $G(V_n)$. Therefore $\gamma\left(G(V_n)\right) = |D_c| = 1$.

Obviously the induced subgraph $\langle D_c \rangle$ is connected. Hence D_c becomes a connected dominating set of $G(V_n)$ with minimum cardinality.

Hence $\gamma_c\left(G(V_n)\right) = 1$. ■

Theorem 5: If n is the product of two distinct primes, then the connected domination number of $G(V_n)$ is 1.

Proof: Suppose $n = p_1 p_2$ where p_1 and p_2 are two distinct primes. Then the $G(V_n)$ graph contains three vertices viz., p_1, p_2 and $p_1 p_2$.

Since GCD $(p_1, p_1 p_2) = p_1$ and GCD $(p_2, p_1 p_2) = p_2$, it follows that vertex $p_1 p_2$ is adjacent to both the vertices p_1 and p_2. Hence if $D_c = \{ p_1 p_2 \}$, then D_c becomes a dominating set of V_n. Therefore $\gamma\left(G(V_n)\right) = 1$.

Obviously the induced subgraph $\langle D_c \rangle$ is connected. Hence D_c becomes a connected dominating set of $G(V_n)$ with minimum cardinality.

Hence $\gamma_c\left(G(V_n)\right) = 1$. ■

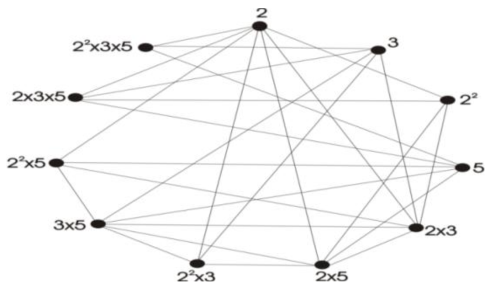
Theorem 6 : If n is neither a prime nor a prime power nor product of two distinct primes, then the connected domination number of $G(V_n)$ is k where k is the core of n.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots. p_k^{\alpha_k}$, where $\alpha_i \geq 1$. To get a connected dominating set of $G(V_n)$, we proceed as follows.

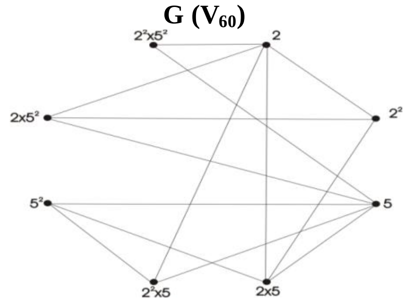
Let $D_c = \{ p_1, p_2, \ldots\ldots, p_{k-1}, t \}$ where $t = p_1 p_2 \ldots\ldots p_k$, the product of k distinct prime divisors of n, which is a vertex in $G(V_n)$. As proved in Theorem 2 , D_c is a minimum dominating set of $G(V_n)$. Also the vertices $p_1, p_2, \ldots\ldots, p_{k-1}$ are adjacent to the vertex t respectively, as GCD $\left(t, p_j\right) = p_j$, for $j = 1, 2, \ldots, k-1$. Thus all the vertices in D_c are adjacent to each other. Hence D_c becomes a minimum connected dominating set of $G(V_n)$.

Therefore $\gamma_c\left(G(V_n)\right) = |D_c| = k$. ■

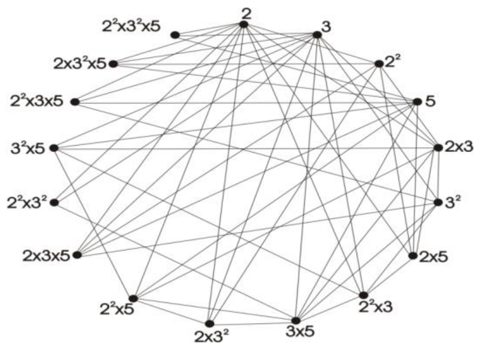
VI. Illustrations



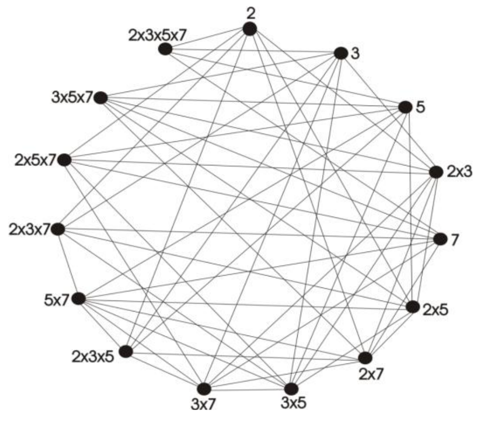
G (V_{60})



G (V_{100})



G (V_{180})



G (V_{210})

$G(V_n)$	$n = 60$ $=2^2 \times 3 \times 5$	$n = 100$ $= 2^2 \times 5^2$	$n = 180$ $= 2^2 \times 3^2 \times 5$	$n = 210$ $= 2 \times 3 \times 5 \times 7$
Minimum Dominating Set	{2,15}	{2,5}	{2,3,5}	{2,3,35}
$\gamma\big(G(V_n)\big)$	2	2	3	3
Minimum Total Dominating Set	{2,3,30}	{2,10}	{2,3,30}	{2,3,5,210}
$\gamma_t\big(G(V_n)\big)$	3	2	3	4
Minimum	{2,15}	{2,5}	{2,3,5}	{2,3,35}

Independent Dominating Set				
$\gamma_i(G(V_n))$	2	2	3	3
Minimum Connected Dominating Set	{2,3,30}	{2,10}	{2,3,30}	{2,3,5,210}
$\gamma_c(G(V_n))$	3	2	3	4

References

[1]. Nathanson, Melvyn B. - Connected components of arithmetic graphs, *Monat. fur. Math, 29,* 1980. 219 – 220.

[2]. Vasumathi, N. - *Number theoretic graphs*, Ph. D. Thesis, S.V.University, Tirupati, India, 1994.

[3]. Berge, C. - *The Theory of Graphs and its Applications*,(Methuen, London) , 1962.

[4]. Ore, O. - Theory of Graphs, *Amer. Math. Soc. Colloq. Publ., 38,* Providence, 1962.

[5]. Haynes, T. W., Hedetniemi, S.T., Slater, P.J. - *Domination in graphs: Advanced Topics* (Marcel Dekker, Inc., New York), 1998.

[6]. Haynes, T.W., Hedetniemi, S.T., Slater, P.J. - *Fundamentals of domination in graphs* (Marcel Dekker, Inc., New York) , 1998.

[7]. Cockayne, C.J. , Dawes, R.M., Hedetniemi, S.T. - Total domination in graphs, *Networks, 10,* 1980, 211-219.

[8]. Allan, R.B., Laskar, R.C., Hedetniemi, S.T. - A note on total domination, *Discrete Math. 49*, 1984, 7-13 .

[9]. Allan, R.B, Laskar, R.C. - On domination and independent domination numbers of a graph, *Discrete Math, 23,* 1978, 73-76.

[10]. Sampathkumar, E., Walikar, H.B. - The connected domination number of a graph, *Math.Phys. Sci.,13*, 1979, 607-613.

[11]. Hedetniemi, S.T., Laskar, R.C., - Connected domination in graphs, *Graph Theory and Combinatorics* (Academic Press, London, 1984).