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On The Sublattice [I, C0] Of the Lattice of Cech Closure Operators

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Abstract: *The interval [I,C_0] where I is the indiscrete closure operator and C_0 is the co-finite closure operator on a set X is a complete sublattice of the lattice of all closure operators on X. In this paper, we determine a class of automorphisms of the lattice [I,C_0] and characterize the group of automorphisms on [I,C_0] when X is finite.*

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I. Introduction

In 1958, Juris Hartmanis [4] determined the automorphisms of the lattice LT(X) of all topologies on a fixed set X as follows : for $p \in S(X)$ and $\tau \in LT(X)$, define the mapping A_p by $A_p(\tau) = \{ p(U) : U \in \tau\}$. Then $A_p(\tau)$ is a topology on X and $A_p(\tau)$ is an automorphism of LT(X). If X is infinite or X contains atmost two elements, the group of automorphisms of LT(X) is precisely $\{A_p : p \in S(X)\}$. Otherwise, the group of automorphisms of LT(X) is $\{A_p: p \in S(X)\} \cup \{B_p : p \in S(X)\}$ where $B_p : LT(X) \to LT(X)$ is defined by $B_p(\tau) = \{ X\text{-}p(U) : U \in \tau\}$ for $\tau \in LT(X)$. From this result, we can conclude that, if X is an infinite set and P is any topological property, then the set of topologies in LT(X) possessing the property P may be identified simply from the lattice structure of LT(X), since the only automorphisms of LT(X) for infinite X are those which simply permute elements of X. Therefore any automorphism of LT(X) must map all the topologies in LT(X) onto their homeomorphic images. Thus the topological properties of elements of LT(X) must be determined by the position of the topologies in LT(X). In this paper, we determine a class of automorphisms of the lattice [I, C_0] and characterize the group of automorphisms on [I,C_0] when X is finite.

II. Preliminaries

Definition 2.1

Let $\wp(X)$ denotes the power set of a set X. A Cech closure operator on a set X is a function $V : \wp(X) \to \wp(X)$ such that,

(i) $V(\phi) = \phi$

(ii) $A \subseteq V(A)$ for all $A \in \wp(X)$

(iii) $V(A \cup B) = V(A) \cup V(B)$ for all $A,B \in \wp(X)$.

For brevity, we call V a closure operator on X and the pair (X,V) a closure space.

Definition 2.2

Let (X,V) be a closure space. A subset A of X is said to be closed, if V(A) = A and open, if V(X - A) = X - A.

Definition 2.3

The set of all open sets in (X,V) is a topology on X, called the topology associated with V. On the other hand, to every topology τ on X, we can associate a closure operator V on X (the Kuratowski closure operator) defined by V(A) = cl(A) where cl(A) denotes the closure of A in (X,τ). We say that V is the closure operator associated with τ.

Remark 2.4

A closure operator on a set need not be the closure operator associated with the topology associated with it. In this sense Cech closure operators on a set X can be considered as generalization of topologies on X.

Definition 2.5

Let V_1 and V_2 be two closure operators on a set X. Then V_1 is said to " coarser than " V_2 (or V_2 is said to finer than V_1) if $V_1(A) \supseteq V_2$ for all $A \in \wp(X)$. In this case we write $V_1 \leq V_2$.

Example 2.6

Let $V : \wp(X) \to \wp(X)$ be defined by

$$V(A) = \phi \text{ if } A = \phi$$
$$= X \text{ otherwise.}$$

Then V is a closure operator on X, called the indiscrete closure operator. The indiscrete closure operator is usually denoted by I.

Example 2.7

Let X be an infinite set. Define $V : \wp(X) \to \wp(X)$ by

$V(A) = A$ if A is finite,

$= X$ otherwise.

Then V is a closure operator on X, called the co-finite closure operator. The co-finite closure operator is usually denoted by C_0 .

Remark 2.8

The relation " coarser than " is a partial order on the set of all closure operators on X. We denote the set of all closure operators on a set X by LC(X). Then LC(X) is a complete lattice under the relation "coarser than " and the least element of this lattice is I.

Definition 2.9

A closure operator on X other than I is called an infra closure operator, if the only closure operator on X strictly smaller than it is I, the indiscrete closure operator on X. Note that the infra closure operators on X are precisely the atoms of the lattice LC(X).

Definition 2.10

For $a,b \in X$, $a \neq b$, define $V_{(a,b)}$ by,

$V_{(a,b)}(A) = A$ if $A = \phi$,

$= X- \{b\}$ if $A = \{a\}$,

$= X$ otherwise.

Then $V_{(a,b)}$ is an infra closure operator on X.

Theorem 2.11

A closure operator on X is an infra closure operator if and only if it is of the form $V_{(a,b)}$ for some $a,b \in X$ such that $a \neq b$. **[6]**

Notation 2.12

We use the notation Ω to denote the atoms of the lattice LC(X) . Then by the Theorem 1, the members of Ω are of the form $V_{(a,b)}$ where $a,b \in X, a \neq b$.

Remark 2.13

The interval $[I,C_0]$ where I is the indiscrete closure operator and C_0 is the co-finite closure operator on a set X is a complete sublattice of the lattice LC(X).

Remark 2.14

The set of all closure operators $[I, C_0]$ on a set X under the partial order " $\leq$ " defined by $V_1 \leq V_2 \Leftrightarrow V_2(A) \subseteq V_1(A)$ for every $A \in \wp(X)$ is a complete lattice.

Notation 2.15

We use the notation S(X) to denote the set of all bijections on X.

III. Main Results

Lemma 3.1

For a closure operator V on X such that $V \leq C_0$, define a relation ρV on X by $\rho V = \{ (x,y) : y \in V(\{x\}) \}$. Then ρV is a reflexive relation on X.

Proof : Obvious.

Lemma 3.2

For $R \in LR(X)$, the lattice of all reflexive relation on X, define $\upsilon R : \wp(X) \to \wp(X)$ by $\upsilon R(A) = \{ y \in X : xRy$ for some $x \in A\}$, $A \in \wp(X)$. Then υR is a closure operator in $[I,C_0]$.

Proof: Obvious.

Remark 3.3

It can be easily verified that, the mapping $\upsilon : LR(X) \to [I, C_0]$ defined by $\upsilon(R) = \upsilon R$ and the inverse mapping $\rho : [I, C_0] \to LR(X)$ defined by $\rho(V) = \rho V$ are dual isomorphisms.

Theorem 3.4

Let X be a non-empty set. For $V \in [I, C_0]$ and $p \in S((X \times X) - \Delta))$, let $R_{p,V} = p(\rho V - \Delta) \cup \Delta$. Then $R_{p,V} \in LR(X)$. Further, let $T_pV = \upsilon R_{p,V}$. Then $T_pV \in [I, C_0]$ and the mapping T_p defined by $T_p(V) = T_pV$ for $V \in [I, C_0]$ is an automorphism of $[I, C_0]$.

Proof:

From the Lemmas 3.1, 3.2 & Remark 3.3, it follows that for $V_1 , V_2 \in [I,C_0]$,

$$V_1 \leq V_2 \Leftrightarrow \rho V_2 \subseteq \rho V_1$$
$$\Leftrightarrow R_{p,V2} \subseteq R_{p,V1}$$
$$\Leftrightarrow T_pV_1 \leq T_pV_2$$
$$\Leftrightarrow T_p(V_1) \leq T_p(V_2)$$

Further, since the correspondences $V \to \rho V$, $\rho V \to \rho V - \Delta$, $\rho V - \Delta \to p(\rho V - \Delta)$, $p(\rho V - \Delta) \to p(\rho V - \Delta) \cup \Delta$ ($= R_{p,V}$) and $R_{p,V} \to T_p V$ are bijections, it follows that $T_p : V \to T_p V$ is a bijection.
Hence T_p is an automorphism of the lattice $[I,C_0]$.

Remark 3.5

Obviously the set of atoms of the lattice $[I,C_0]$ is precisely the set $\Omega = \{ V_{(a,b)} : a,b \in X \text{ and } a \neq b \}$.

Theorem 3.6

Let X be a non-empty finite set. Then the lattice $[I,C_0]$ coincides with the lattice LC(X) of all Cech closure operators on X and hence the group of the lattice $[I,C_0]$ is precisely the set $\{T_p : p \in S((X \times X) - \Delta)\}$.

Proof:

Let A be any automorphism of the lattice $[I,C_0]$. We want to show that $A = T_p$ for some $p \in S((X \times X) - \Delta)$. For $V_{(a,b)} \in \Omega$, let $A(V_{(a,b)}) = V_{(a,b)'}$ for some $(a,b)' \in (X \times X) - \Delta$. Then $(a,b)'$ is unique. Define $p(a,b) = (a,b)'$. Then $p \in S((X \times X) - \Delta)$.

Now for $V_{(a,b)} \in \Omega$,

$$\begin{aligned} A(V_{(a,b)}) &= V_{(a,b)'} \\ &= \upsilon [(\rho V_{p(a,b)} - \Delta) \cup \Delta] \\ &= \upsilon [p (\rho V_{(a,b)} - \Delta) \cup \Delta] \\ &= T_p V_{(a,b)} \\ &= T_p (V_{(a,b)}) \end{aligned}$$

Hence $A = T_p$ on Ω. Since X is finite, the lattice $[I,C_0] =$ LC(X) is atomistic and hence it follows that $A = T_p$ on $[I,C_0]$.

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