IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)
e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 16, Issue 4 Ser. I (Jul. - Aug. 2019), PP 01-16
www.iosrjournals.org

Some problems with numerical calculations of the meteo-ballistic sensitivity functions and their solutions

Vladimir Cech[1], Jiri Jevicky[2]
[1](Department of Weapons and Ammunition, University of Defence, Kounicova 65, 662 10 Brno, Czech Republic)
[2](Department of Weapons and Ammunition, University of Defence, Kounicova 65, 662 10 Brno, Czech Republic)

Abstract: *Projectile trajectories calculated under non-standard conditions are considered to be perturbed. Tools utilized for the analysis of perturbed trajectories are sensitivity functions: effect functions, weighting factor functions (WFFs) and appropriate Green's functions. These functions are used for calculation of meteo ballistic elements μ_B (ballistic wind vector $\boldsymbol{w}_B$, virtual temperature τ_B, density ρ_B) as well.*
Since 2013 we are working on improving the theory of these functions. We published the improved theory of generalized meteo-ballistic WFFs in the journal Defence technology in the year 2016 and then the improved theory of projectile trajectory reference heights in the year 2017. Using these theories will improve methods for designing firing tables, fire control systems algorithms, and meteo message generation algorithms.
This contribution complements the previous two articles and is dedicated to the key problems of numerical calculations of the sensitivity functions.

Keywords*: Exterior ballistic; sensitivity analysis; non-standard (perturbed) projectile trajectory; Green's function; Weighting factor function (curve); effect function; norm effect*

Date of Submission: 22-06-2019 Date of acceptance: 05-07-2019

I. Introduction

This contribution follows up on our earlier publications [1 – 7]. For the sake of understanding the contents this article, it is, at least, needful to peruse problems of weighting (factor) functions (curves) WFFs presented in [4, 6]. The traditional theory of the reference height of a trajectory RHT is elaborated in the article [1] predominatinglyunderutilization of the textbook [8]. Our new improved theory of projectile trajectory RHT is presented in [6].

1.1 *Motivation*

We continue the research with the same theme and therefore our motivation cannot change [1 – 7]: It follows from the analysis of artillery fire errors, e.g. [8, 9], that approximately two-thirds of the inaccuracy of indirect artillery fire is caused by inaccuracies in the determination of meteo parameters included in the error budget model [9]. Consequently, it is always important to pay close attention to the problems of including the actual meteo parameters in ballistic calculations [8]. The following meteo parameters μ are primarily utilized: Wind vector w, air pressure p, virtual temperature τ, density ρ, and speed of sound a [8, 10 – 18].
This paper deals only with problems relating to unguided projectiles without propulsion system for the sake of lucidity of the solved problems.

1.2 *References*

The whole theory of meteo-ballistic sensitivity functions is based on the theory of perturbations [19]. Its application in aeromechanics and external ballistics can be found for example in [10 – 16, 18, 20 – 23].

For solutions of perturbed tasks the Green's functionsare very often used, for example [11, 13, 16, 22, 23]. Only in [22] it is explicitly claimed that these are the Green's functions. Other authors use them as "nameless functions".

In external ballistics there are traditionally used [8, 10, 11, 13 – 16] effect functions, weighting factor functions (WFFs) and their derivatives instead of the corresponding Green's functions [1 – 7].
The perturbation theory is closely connected with the theory of sensitivity of dynamic systems [21, 24]. The best way to analyze the characteristics of the projectiles trajectories under nonstandard conditions is the build of any of the explicit sensitivity models of projectile trajectory [21, 23, 25, 26]. We then speak about the (differential) sensitivity analysis of dynamical system (projectile trajectory) or about the parameter sensitivity analysis or about the sensitivity of a system to parameter variations.Without this theory, firing tables cannot be compiled and algorithms for the formation of meteorological messages cannot be created [8, 10, 12, 15 – 18, 27 - 31].In some books, for example [25, 26], only brief remarks about these problems can be found.

List of the notation

μ	met parameter (element)
$\mu(y)$	real or measured magnitude of met parameter μ in height y
$r(\mu)$	weighting factor function (curve, WFF)
Q_P, Q_{CP}	effect function
$\mu_{STD}(h)$	met parameter standard course with the height h
$\Delta\mu(y)$	absolute deviation of met element μ in height y
$\delta\mu(y)$	relative deviation of met element μ in height y
$\Delta\mu_B$	absolute ballistic deviation of ballistic element μ_B
$\delta\mu_B$	relative ballistic deviation of ballistic element μ_B

1.3. ***The main objectives of the contribution***

We published the improved theory of generalized meteo-ballistic WFFs in the journal Defence technology in the year 2016 [4] and then the improved theory of projectile trajectory reference heights in the year 2017 [6]. Using these theories will improve methods for designing firing tables, fire control systems algorithms, and meteo message generation algorithms.

The permissible range of articles did not allow us to deal with the problem of the numerical calculations of the sensitivity functions. Therefore, this article complements the previous two articles and is dedicated to the key problems of numerical calculations of these functions.

The aim of this article is to offer a numerical solution of the following problems:

1. Efficient calculations of the non-isochronous effect functions – the section 2.
2. Creating the Garnier’s and Bliss‘ notations of Green’s functions – the section 3.
3. To derive a new calculation relationship for the generalized reference height of trajectory – the subsection 4.1.
4. To derive new calculation relationships for the ballistic perturbations/deviations $\Delta\mu_B$ and $\delta\mu_B$ – the subsection 4.2.

An efficient calculation of the non-isochronous effect functions (the section 2) presupposes the finding of the decision-making criteria for the completion of the integration of partially perturbed projectile trajectory [6] – the relation (8). Simultaneously, we derive relations for the approximate conversion values of an isochronous effect function on values of non-isochronous one (the subsection 2.2, relations (16), (17)), and subsequently we will explain the problem [6] of exact and strong effects (the subsection 2.3).

During the creation of the Garnier’s and Bliss‘ notations [4] of Green’s functions (the section 3) arises the problem of division by zero (the subsection 3.4). We suggest how this complication can be circumvented.

We derive a new calculation relationship for the generalized reference height of trajectory (RHT, the subsection 4.1, the relation (58)), which bypasses the need for numerical calculation of the first derivative of the WFF, determined by the table of its discrete values [6].

We derive new computational relations for the ballistic perturbations/deviations $\Delta\mu_B$ and $\delta\mu_B$ (the subsection 4.2, relations (62), (67)), when we also bypass the need for numerical calculations of the first derivative of the WFF, which is determined by the table of its discrete values [6].

II. Effect And Green’s Functions

In the whole section we assume, that to calculate the WFFs and Green’s functions, we use the implicit Garnier's algorithm, the principle of which we have clarified in subsections 1.5 and 2.2 of our contribution [4]. The algorithm is particularly suitable for the use on fast digital computers. Its main advantage is that it works only with the mathematical model of the projectile trajectory (3 or 4 or 6 degree of freedom – DOF) and it is not necessary to create neither an appropriate perturbation model nor a sensitivity model of the projectile trajectory. The outputs of the Garnier’s algorithm are the relevant effect functions in the time domain, from which the estimates of the WFFs and Green’s functions are calculated numerically in the second step, again in the time domain [4].

2.1 ***Definition of Effect functions and Green’s functions in time domain***

The perturbation theory is often used to create appropriate sensitivity models [19, 21, 24]. These are the linearized models represented by systems of linear differential equations with variable coefficients. In the case of Garnier’s algorithm this system also exists, but is not presented in the explicit form.

Relations [4, 6] between generalized inputs (control input variables, disturbance input variables and variable parameters of the system) $z_m(t)$, $m = 1, 2, ...$ on the one hand and output variables y_l, $l = 1, 2, ...$ on the other hand, are given traditionally by transfer functions and Green's functions $g_{m,l}(t - t_P)$ or effect functions $Q_{m,l}(t - t_P)$. There is also a generalized theory of Green's functions for some groups of non-linear systems.

In our case [4, 6], we take into consideration only the following generalized inputs: the wind vector $\boldsymbol{w} = (w_x, w_y, w_z)$ as a disturbance input variable and next variable meteo parameters (the virtual temperature τ, the air pressure p, the air density ρ and the speed of sound a). All we denote, as we have already noted, by the common symbol μ. In the case of other parameters of the model of the projectile trajectory, it progresses as well.

The most important output variables are the coordinates of the partially perturbed projectile trajectory $(x, y, z)_P$ and corresponding time of flight t.

Green's functions are also denoted as weight or weighting functions or influence functions or impulse response functions. In the case of sensitivity models of a dynamical system Fig. 1, the Green's functions and effect functions represent special sensitivity functions of two parameters (t, t_P):

t is the moment to which the system response will be calculated. In our case, it is the moment of t_{PI} in which the standard projectile trajectory $(t, x, y, z)_{STD}$ passes through chosen point of impact/burst PI with coordinates $(t, x, y, z)_{PI}$. According to its position on the projectile trajectory, we distinguish four types of the projectile trajectories ($i = 1, 2, 3, 4$) – Table 2 in [4]. We have introduced the concept of the "basic trajectory" to their definition. The origin of its trajectory is at time $t = 0$, the top ($y = y_{max} = y_S$) of its trajectory is at $t = t_S$ and the point of fall ($y = 0$) in $t = t_F$. The beginning of the i^{th} trajectory is at the time $t = t_{O,i} \geq 0$ and its end in the chosen point of impact (burst) PI at the time respectively $t = t_{PI,i} > t_{O,I}$ and $t_{end,i} > t_{O,i}$. To simplify the notations, we use the relative time $t_r = t - t_{O,i}$. If it does not cause misunderstanding, we will write the relative time t_r without the index „r", then simply „t". The index "r" will be also skipped at other times, such as t_P instead of t_{Pr}, t_{PI} instead of t_{PIr} – Fig. 1, 2.

t_P is the moment in which to impress $z_m(t - t_P) = z_m(t) + (+/–)\ \Delta z_{m0}(t_P)\cdot\varepsilon(t - t_P)$, where $z_m(t)$ is unperturbed quantity, $\Delta z_{m0}(t_P)$ is the amplitude of excitation and the function $\varepsilon(t - t_P)$ is in the case of the calculations of effect functions equal to $H(t - t_p)$ (the Heaviside step function) and in the case of the calculations Green's functions it is equal to $\delta(t - t_p)$ – the unit impulse (the Dirac delta function).The plus sign applies for the calculation of the positive perturbation and the minus sign for the calculation of the negative perturbation.

We distinguish three types of trajectories of the projectile:

- partially perturbed trajectory Fig. 1, if $t_{Pr} \in (0, t_{PI})$ and its special variants:
- (fully) perturbed trajectory, if $t_{Pr} = 0$, i.e. the impulse impresses at the beginning of the i^{th} trajectory ($t_P = t_{O,i}$) and
- the standard (unperturbed) trajectory, if $t_P \geq t_{PI}$.

We calculate perturbations in the common moment $t_{end} = t_{PI} + \Delta t_{PI}$. We consider only the isochronous perturbation ($t_{end} = t_{PI}$, $\Delta t_{PI} = 0$) in this subsection – Fig. 1. The non-isochronous perturbations will be discussed in the subsection 2.2 – Fig. 2.

Perturbations are divided in general into the basic and combined. If only one generalized input $z_m(t)$, $m = 1, 2, \ldots$, is the subject to perturbation, then it is the basic perturbation. If two or more generalized inputs are the subjects to perturbation at the same time t_P, then the combined perturbation it is generated.

The basic perturbations are the most significant for practice. The most important are listed in the Table 1. The most important responses to the basic perturbations – effect functions – are presented in the Table 2.

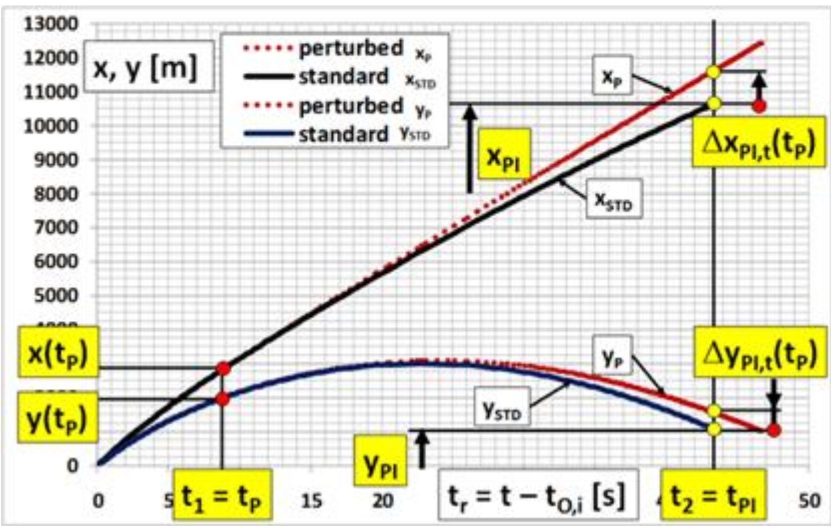

Fig. 1 Partially perturbed trajectory $(x_P(t_P), y_P(t_P))$ and its isochronous perturbations ($\Delta t_{PI,t} = 0$) $\Delta x_{PI,t}(t_P)$, $\Delta y_{PI,t}(t_P)$ in the point of impact (burst) PI $(x(t_{PI}), y(t_{PI}))_{STD} = (x_{PI}, y_{PI})_{STD}$.

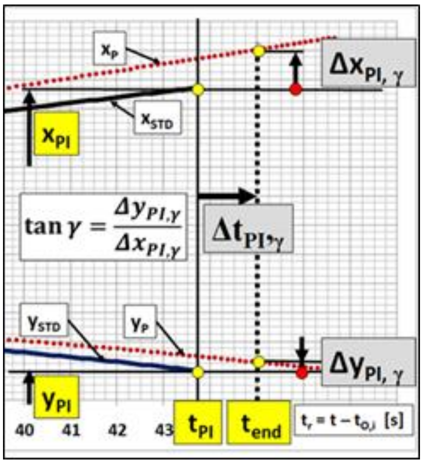

Fig. 2 Zoom of the Fig. 1 and non-isochronous perturbations ($t_{\mathrm{end}} = t_{\mathrm{PI}} + \Delta t_{\mathrm{PI},\gamma}$) $\Delta x_{\mathrm{PI},\gamma}(t_{\mathrm{P}})$, $\Delta y_{\mathrm{PI},\gamma}(t_{\mathrm{P}})$ in the point of impact (burst) PI.

Table 1

m	$z_m(t) = z_{m,\mathrm{STD}}(t) = \mu_{\mathrm{STD}}(t)$	$\Delta z_{m0}(t_{\mathrm{P}}) = \Delta\mu(t_{\mathrm{P}})$	$N_{zm} = N_{\mu} = \mathrm{const} = (\Delta\mu_{\mathrm{B0}}$ or $\delta\mu_{\mathrm{B0}})$
1	0	w_{x0}	w_{x0}
2	0	w_{y0}	w_{y0}
3	0	w_{z0}	w_{z0}
4	$\tau_{\mathrm{STD}}(t)$	$\Delta\tau_0$	$\Delta\tau_0$
5	$\tau_{\mathrm{STD}}(t)$	$\Delta\tau(t_{\mathrm{P}}) = \delta\tau_0\cdot\tau_{\mathrm{STD}}(t_{\mathrm{P}})$	$\delta\tau_0$
6	$p_{\mathrm{STD}}(t)$	$\Delta p(t_{\mathrm{P}}) = \delta p_0\cdot p_{\mathrm{STD}}(t_{\mathrm{P}})$	δp_0
7	$\rho_{\mathrm{STD}}(t)$	$\Delta\rho(t_{\mathrm{P}}) = \delta\rho_0\cdot\rho_{\mathrm{STD}}(t_{\mathrm{P}})$	$\delta\rho_0$

Table 2

l	**Standard parameters**	**Partiallyperturbedparameters**	**Effectfunctions**
	$y_l(t_{\mathrm{PI},i}) = y_{l,\mathrm{STD}}$	$y_l(\Delta z_{m0}(t_{\mathrm{P}}), t_{\mathrm{P}}, t_{\mathrm{end}}) = y_{l,\mathrm{end}}$	$Q_{\mathrm{P}} = Q_{\mathrm{P}}(t_{\mathrm{P}}) = Q_{\mathrm{P}}(N_{\mu}, t_{\mathrm{P}}) = y_{l,\mathrm{end}} - y_{l,\mathrm{STD}}$
1	$t_{\mathrm{PI}} = t_{\mathrm{PI},i}$	t_{end}	$\Delta t_{\mathrm{PI}} = \Delta t_{\mathrm{PI},i}$
2	x_{PI}	x_{end}	$\Delta x_{\mathrm{PI}} = \Delta x_{\mathrm{PI},i}$
3	y_{PI}	y_{end}	$\Delta y_{\mathrm{PI}} = \Delta y_{\mathrm{PI},i}$
4	z_{PI}	z_{end}	$\Delta z_{\mathrm{PI}} = \Delta z_{\mathrm{PI},i}$
5	$v_{x,\mathrm{PI}}$	$v_{x,\mathrm{end}}$	$\Delta v_{x,\mathrm{PI}} = \Delta v_{x,\mathrm{PI},i}$
6	$v_{y,\mathrm{PI}}$	$v_{y,\mathrm{end}}$	$\Delta v_{y,\mathrm{PI}} = \Delta v_{y,\mathrm{PI},i}$
7	$v_{z,\mathrm{PI}}$	$v_{z,\mathrm{end}}$	$\Delta v_{z,\mathrm{PI}} = \Delta v_{z,\mathrm{PI},i}$
8	v_{PI}	v_{end}	$\Delta v_{\mathrm{PI}} = \Delta v_{\mathrm{PI},i}$
9	Θ_{PI}	Θ_{end}	$\Delta\Theta_{\mathrm{PI}} = \Delta\Theta_{\mathrm{PI},i}$
10	ψ_{PI}	ψ_{end}	$\Delta\psi_{\mathrm{PI}} = \Delta\psi_{\mathrm{PI},i}$
11	etc.	etc.	etc.

Comment: Ground speed vector $\boldsymbol{v} = (v_x, v_y, v_z)$, $v_x = v\cdot\cos\Theta\cdot\cos\psi$, $v_y = v\cdot\sin\Theta$, $v_z = v\cdot\cos\Theta\cdot\sin\psi$.

The norms for the generalized inputs $N_{zm} = N_{\mu} = (\Delta\mu_{\mathrm{B0}}$ or $\delta\mu_{\mathrm{B0}}) = \mathrm{const} > 0$ are listed in the last column of the Table 1. In the case of absolute perturbations, it is $N_{zm} = N_{\mu} = \Delta\mu_{\mathrm{B0}} > 0$; we used in relevant expressions index "A" (absolute) in previous articles [1 – 7]. In the case of relative perturbations, it is $N_{zm} = N_{\mu} = \delta\mu_{\mathrm{B0}} > 0$; we used in relevant expressions index "R" (relative) in previous articles [1 – 7]. See more closely in [4] □ commentaries to the relation (9) and to the Table 1.

Because we use the Garnier's algorithm, the effect functions are directly listed in the last column of the Table 2. We will use the abbreviated designation Q_{P} or $Q_{\mathrm{P}}(t_{\mathrm{P}})$ or $Q_{\mathrm{P}}(N_{\mu}, t_{\mathrm{P}})$ for these functions in the further text. See also the subsection 2.2 in [4] for their calculation.

Besides the calculated values Q_{P} of the effect functions, their normed (standardized) shapes are also used. Three ways of normalization are used in practice [4, 6, 8, 10, 14 – 17, 27]:

a) Unit effect functions

$$R_{\mathrm{P}} = R_{\mathrm{P}}(t_{\mathrm{P}}) = R_{ml}(t_{\mathrm{P}}) = \frac{1}{N_{\mu}}\cdot Q_{\mathrm{P}}(N_{\mu}, t_{\mathrm{P}}). \qquad (1)$$

b) Weighting factor functions (curves) – WFFs

$$r = r(t_{\mathrm{P}}) = r(N_{\mu}, t_{\mathrm{P}}) = \frac{1}{\sigma_{\mathrm{Q}} \cdot N_{\mathrm{Q}}} \cdot Q_{\mathrm{P}}(N_{\mu}, t_{\mathrm{P}}), \qquad (2)$$

where
$N_{\mathrm{Q}} = N_{\mathrm{QN}} \cdot N_{\mu}$ is the norm for WFF,
$\sigma_{\mathrm{Q}} = -1$ or $+1$.
The detailed clarification of the choice of the values N_{Q} a σ_{Q} see in [6], subsection 1.4.2, and also in [4], subsection 2.5.4.

b) Unit effect factor and unit correction factor

$$Q_{\mathrm{P,S}} = Q_{\mathrm{P,S}}(t_{\mathrm{P}}) = Q_{\mathrm{P,S}}(N_{\mu}, t_{\mathrm{P}}) = \sigma_{\mathrm{S}} \cdot \left(\frac{N_{\mu\mathrm{B}}}{N_{\mu}}\right) \cdot Q_{\mathrm{P}}(N_{\mu}, t_{\mathrm{P}}), \quad (3)$$

where
$\sigma_{\mathrm{S}} = +1$ is for the unit effect factor,
$\sigma_{\mathrm{S}} = -1$ is for the unit correction factor,
$N_{\mu\mathrm{B}}$ =const = ($\Delta\mu_{\mathrm{BN}}$ or $\delta\mu_{\mathrm{BN}}$) > 0 are new standardizing values. The value of $\Delta\mu_{\mathrm{BN}}$ corresponds to $\Delta\mu_{\mathrm{B0}}$ and the value $\delta\mu_{\mathrm{BN}}$ to $\delta\mu_{\mathrm{B0}}$ – Table 1. More closely in [4]: a comment to the relation (9) and the subsection 1.4.
The selected values $Q_{\mathrm{P,S}}$ (Table 2, rows 1, 2, 3, 4) calculated for the fully perturbed trajectory ($t_{\mathrm{P}} = t_{O,i}$) are given in the tabular firing tables [4, 6, 8, 10, 16 – 18, 27].
The Green's functions are defined implicitly

$$R_{\mathrm{P}}(t_{\mathrm{P}}) = \int_{t_{\mathrm{P}}}^{t_{\mathrm{PI}}} g(t'_{\mathrm{P}}) \cdot \mathrm{d}t'_{\mathrm{P}}, \qquad (4)$$

and so explicitly (does not apply at points where the derivative of the function is not continuous)

$$g(t_{\mathrm{P}}) = -\frac{\mathrm{d}R_{\mathrm{P}}}{\mathrm{d}t_{\mathrm{P}}} = -\left(\frac{\sigma_{\mathrm{Q}} \cdot N_{\mathrm{Q}}}{N_{\mu}}\right) \cdot \frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{P}}}. \qquad (5)$$

The further comments on relations (4), (5), (6) are in [4] and a comment to the relation (1) is in [6].
Absolute ballistic deviation/perturbation $\Delta\mu_{\mathrm{B}}$ of the ballistic meteo parameter μ (e.g. ballistic range/cross wind $(w_x, w_z)_{\mathrm{B}}$, absolute ballistic virtual temperature deviation/perturbation τ_{B}, etc.).

$$\Delta\mu_{\mathrm{B}} = -\int_{t_{\mathrm{O},i}}^{t_{\mathrm{P},i}} \Delta\mu(t'_{\mathrm{P}}) \cdot \left(\frac{\mathrm{d}r(\mu_{\mathrm{B0}}, t_{\mathrm{P}})}{\mathrm{d}t_{\mathrm{P}}}\right) \cdot \mathrm{d}t'_{\mathrm{P}} = \left(\frac{N_{\mu}}{\sigma_{\mathrm{Q}} \cdot N_{\mathrm{Q}}}\right) \cdot \int_{t_{\mathrm{O},i}}^{t_{\mathrm{PI},i}} g(\Delta\mu_{\mathrm{B0}}, t'_{\mathrm{P}}) \cdot \mathrm{d}t'_{\mathrm{P}}, \quad (6)$$

where $\Delta\mu(t) = \mu(t) - \mu_{\mathrm{STD}}(t)$ is the absolute deviations, $\mu(t)$ – measured (given) values.
Analogous relations to (6) also applies for the known (measured) relative deviations $\delta\mu(t)$ ($\delta\mu(t) = \Delta\mu(t)/\mu_{\mathrm{STD}}(t)$) [4, 6, 8, 16].

2.2 *Non-isochronous perturbations*
In practice, preference is given to non-isochronous perturbations from isochronous perturbations [10, 12, 14 – 18, 27].

It is generally assumed that the perturbed trajectory crosses at the moment t_{end} to surface $F(x,y,z) = 0$, in which lies the point of impact (burst) PI. The relevant iso-surface perturbations are calculated from the corresponding coordinates – Fig. 2. This surface may represent the ground surface or the surface of a large target in the neighborhood of the point PI.

In practice, the easiest surface is used, and that is the plane which is perpendicular to the plane (x, y) [4, 10 – 17], so its location is determined by the angle of inclination γ – Fig. 3. We determine, therefore, the iso-planar perturbations.
Our task is to derive a rule for termination of the calculation of the perturbed trajectory just in time t_{end}. It holds in general (Fig. 3, Table 2)

$$\tan\gamma = \frac{y_{\mathrm{PI},\gamma}}{x_{\mathrm{PI},\gamma}} = \mathrm{const}\,. \qquad (7)$$

Using this relationship, we can create the function

$$F_{\mathrm{end}}(t) = \left[x_{\mathrm{P}}\left(N_{\mu}, t_{\mathrm{P}}, t\right)\cdot\sin\gamma - y_{\mathrm{P}}\left(N_{\mu}, t_{\mathrm{P}}, t\right)\cdot\cos\gamma\right] - F_{\mathrm{end},0}\,, \qquad (8)$$

where

$$F_{\mathrm{end},0} = \left[x_{\mathrm{PI}}\cdot\sin\gamma - y_{\mathrm{PI}}\cdot\cos\gamma\right] \; .(9)$$

After each integration step, we test whether it is $F_{\mathrm{end}}(t) = 0$, if yes, then $t = t_{\mathrm{end}}$ and the calculation ends. In the case of isochronous perturbation, it applies explicitly, that $t_{\mathrm{end}} = t_{\mathrm{PI}}$.

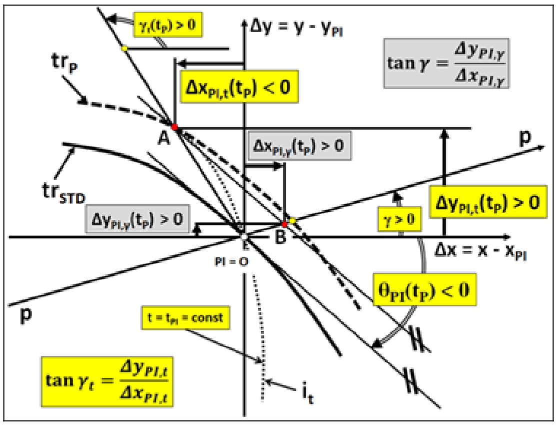

Fig. 3 The schema for the estimation of non-isochronous perturbations ($t_{\mathrm{end}} = t_{\mathrm{PI}} + \Delta t_{\mathrm{PI},\gamma}$) $\Delta x_{\mathrm{PI},\gamma}(t_{\mathrm{P}})$, $\Delta y_{\mathrm{PI},\gamma}(t_{\mathrm{P}})$ in the point of impact (burst) PI from the given isochronous perturbations ($\Delta t_{\mathrm{PI,t}} = 0$) $\Delta x_{\mathrm{PI},t}(t_{\mathrm{P}})$, $\Delta y_{\mathrm{PI},t}(t_{\mathrm{P}})$. $\mathrm{tr}_{\mathrm{STD}}$ – standard trajectory, tr_{P} – perturbed trajectory, p – plane footprint, i_t – isochrone.

Table 3

	γ	$\Delta t_{\mathrm{PI},\gamma}$	$\Delta x_{\mathrm{PI},\gamma}$	$\Delta y_{\mathrm{PI},\gamma}$	$\Delta R_{\mathrm{PI},\gamma}$	**Perturbation case**
1	γ_{t}	0	$\Delta x_{\mathrm{PI,t}}$	$\Delta y_{\mathrm{PI,t}}$	$\Delta R_{\mathrm{PI,t}}$	isochronous perturbation
2	0°	$\Delta t_{\mathrm{PI,Y}}$	$\Delta x_{\mathrm{PI,Y}}$	0	$\Delta x_{\mathrm{PI,Y}}$	iso-height y_{PI}perturbation
3	90°	$\Delta t_{\mathrm{PI,X}}$	0	$\Delta y_{\mathrm{PI,X}}$	$\Delta y_{\mathrm{PI,X}}$	iso-range x_{PI}perturbation
4	$\varepsilon_{\mathrm{PI}}$	$\Delta t_{\mathrm{PI},\varepsilon}$	$\Delta x_{\mathrm{PI},\varepsilon}$	$\Delta y_{\mathrm{PI},\varepsilon}$	$\Delta R_{\mathrm{PI},\varepsilon}$	iso-angle-of-site $\varepsilon_{\mathrm{PI}}$perturbation
5	$\varepsilon_{\mathrm{PI}}$ + 90°	$\Delta t_{\mathrm{PI,D}}$	$\Delta x_{\mathrm{PI,D}}$	$\Delta y_{\mathrm{PI,D}}$	$\Delta R_{\mathrm{PI,D}}$	iso-slant range perturbation
6	θ_{PI} + 90°	$\Delta t_{\mathrm{PI,m}}$	$\Delta x_{\mathrm{PI,m}}$	$\Delta y_{\mathrm{PI,m}}$	$\Delta R_{\mathrm{PI,m}}$	minimal $\Delta R_{\mathrm{PI},\gamma} - \Delta R_{\mathrm{PI,m}}$perturbation

Comment: 1. $D_{\mathrm{PI}} = \sqrt{\left[x_{\mathrm{PI}}^2 + y_{\mathrm{PI}}^2\right]}$ – slant range (standard value).

2. $\varepsilon_{\mathrm{PI}} = \sin^{-1}\left(\dfrac{y_{\mathrm{PI}}}{D_{\mathrm{PI}}}\right)$ – angle-of-site (standard value).

3. $\varepsilon_{\mathrm{PI}} = \sin^{-1}\left(\dfrac{R_{\mathrm{PI},D}}{D_{\mathrm{PI}}}\right)$ – perturbation of the angle-of-site.

If we don't need a precise determination of the time t_{end} and the corresponding iso-planar perturbations, we can obtain their estimates using a linear extrapolation – Fig. 2, 3 – as follows.

In the first step we calculate isochronous perturbation ($\Delta x_{\mathrm{PI,t}}$, $\Delta y_{\mathrm{PI,t}}$, $\Delta y_{\mathrm{PI,t}}$, …) – Table 2, and we calculate an estimate of the angle of inclination of the isochrone i_t – Fig. 3

$$\tan\gamma_t = \frac{\Delta y_{\mathrm{PI},t}}{\Delta x_{\mathrm{PI},t}}\,,(10)$$

and the radial isochronous perturbation

$$\Delta R_{\mathrm{PI},t} = \sqrt{\left(\Delta x_{\mathrm{PI},t}^2 + \Delta y_{\mathrm{PI},t}^2\right)} . \qquad (11)$$

Furthermore, we assume that approximately applies (Table 2) $\Delta v_{x,\mathrm{PI}} \approx \Delta v_{y,\mathrm{PI}} \approx \Delta v_{z,\mathrm{PI}} \approx 0$, then next three equations hold approximately

$$\Delta x_{\mathrm{PI},\gamma} \approx \Delta x_{\mathrm{PI},t} + v_{x,\mathrm{PI}} \cdot \Delta t_{\mathrm{PI},\gamma} , \qquad (12)$$

$$\Delta y_{\mathrm{PI},\gamma} \approx \Delta y_{\mathrm{PI},t} + v_{y,\mathrm{PI}} \cdot \Delta t_{\mathrm{PI},\gamma} , \qquad (13)$$

$$\Delta z_{\mathrm{PI},\gamma} \approx \Delta z_{\mathrm{PI},t} + v_{z,\mathrm{PI}} \cdot \Delta t_{\mathrm{PI},\gamma} . \qquad (14)$$

By analogy to (11) we define the radial iso-planar perturbation

$$\Delta R_{\mathrm{PI},\gamma} = \sqrt{\left(\Delta x_{\mathrm{PI},\gamma}^2 + \Delta y_{\mathrm{PI},\gamma}^2\right)} . \qquad (15)$$

We determine the solution of equations (12) and (13) ($t_{\mathrm{end}} = t_{\mathrm{PI}} + \Delta t_{\mathrm{PI},\gamma}$)

$$\Delta t_{\mathrm{PI},\gamma} \approx \frac{\Delta y_{\mathrm{PI},t} \cdot \cos\gamma - \Delta x_{\mathrm{PI},t} \cdot \sin\gamma}{v_{x,\mathrm{PI}} \cdot \sin\gamma - v_{y,\mathrm{PI}} \cdot \cos\gamma} . \qquad (16)$$

After next adjustment ($\cos\psi \approx 1$, Table 2), we obtain the alternative relationship

$$\Delta t_{\mathrm{PI},\gamma} \approx T_{\mathrm{PI},t} \cdot \frac{\sin(\gamma_t - \gamma)}{\sin(\gamma - \theta_{\mathrm{PI}})}, \qquad (17)$$

where

$T_{\mathrm{PI},t} = \dfrac{\Delta R_{\mathrm{PI},t}}{v_{\mathrm{PI}}}$ is the time constant [s].

In the second step, we calculate an estimate of the time t_{end} using formula (16) or (17) and relations (12) to (15), we calculate estimates of the basic iso-planar perturbations.

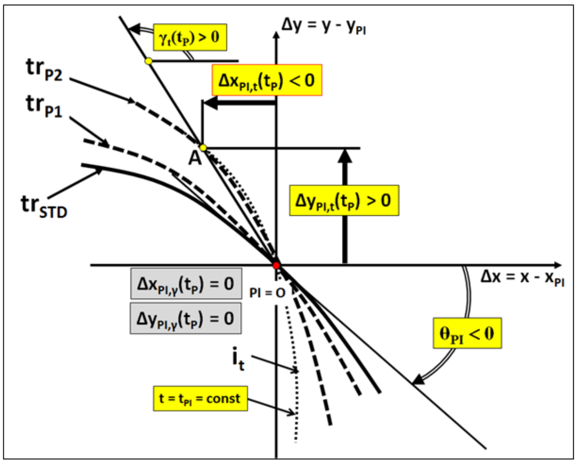

Fig. 4 The schema for explanation of the exact and strong norm effect problem. It's true for exact norm effect $\Delta x_{\mathrm{PI},\gamma}(t_{\mathrm{P}}) = 0$ and $\Delta y_{\mathrm{PI},\gamma}(t_{\mathrm{P}}) = 0$ and for strong norm effect $\Delta x_{\mathrm{PI},\gamma}(t_{\mathrm{P}}) \approx 0$ and $\Delta y_{\mathrm{PI},\gamma}(t_{\mathrm{P}}) \approx 0$; $t_{\mathrm{P}} = t_{\mathrm{O,i}}$, tr_{STD} – standard trajectory, tr_{P} – perturbed trajectory, i_t – isochrone.

In practice, isochronous and iso-planar perturbations are the most commonly used. They are listed in the Table 3. In the tabular firing tables [4, 12, 14, 17, 27], the times $\Delta t_{\mathrm{PI},\gamma}$ are not adduced, but corresponding „fuze setting“ – standard values and unit corrections factors – the relationship (3). More information is in [4], the subsection 2.3.

2.3 *Exact and strong norm effect*

We first discussed the problem of the "norm effect" [15] in [4] (the subsection 2.5.4) and in detail we analyzed it in [6] (the subsection 1.4.2). The geometric point of view on the problem lacked in these analyses. We now explain briefly this approach with the use of Fig. 3, 4.

The problem of the norm's effect applies only to fully perturbed trajectories ($t_{\mathrm{P}} = t_{\mathrm{O},i}$ or $t_{\mathrm{Pr}} = 0$). Even when the norm effect can occur in any of the effect function, historically [15] is discussed for a special selection of effect functions from the Table 2, rows 2 and/or 3, so $\Delta x_{\mathrm{PI}} = \Delta x_{\mathrm{PI},i}$ and/or $\Delta y_{\mathrm{PI}} = \Delta y_{\mathrm{PI},i}$. Even when the norm effect can occur in any of the effect function, further we will consider only the above-defined special case.

Strong norm effect is defined by the following condition (see Fig. 4)

$$\Delta R_{\mathrm{PI},\gamma}\left(t_{\mathrm{O},i}\right) = \sqrt{\left(\Delta x_{\mathrm{PI},\gamma}^{2} + \Delta y_{\mathrm{PI},\gamma}^{2}\right)} \approx 0 \ . \qquad (18)$$

In the case of the exact norm effect, it applies in (18) the exact equality and the traditional method of normalization [4, 6, 15] will be $N_{\mathrm{Q}} = 0$. The consequences are obvious for example from the relations (2), (5), (6).

Using the linearized relations (12), (13) and (17) we perform a brief analysis of the problem.
The Exact norm effect can occur in two cases.
In the first case, except for the relation (18), it holds for the strong norm effect moreover

$$\Delta R_{\mathrm{PI},t}\left(t_{\mathrm{O},i}\right) = \sqrt{\left(\Delta x_{\mathrm{PI},t}^{2} + \Delta y_{\mathrm{PI},t}^{2}\right)} \approx 0 \ . \ (19)$$

In the case of the exact norm effect is true in (19) the exact equality. From the relation (17) immediately implies $\Delta t_{\mathrm{PI},\gamma} \approx 0$, therefore it is a isochronous exact/strong norm effect.

In the second case, the relation (19) does not hold even approximately, so $\Delta t_{\mathrm{PI},\gamma} \neq 0$. Therefore, it is an iso-planar exact/strong norm effect. The determination of the conditions of its occurrence we get them from relations (12) and (13) and from the assumption that the relation (18) is true. After adjustment we obtain the condition

$$\gamma_t \approx \theta_{\mathrm{PI}} \ .(20)$$

From the relations (16) and (17) in this case, an estimate of the size of the $\Delta t_{\mathrm{PI},\gamma}$ cannot be determined. It would be necessary to derive more accurate relationships, which would represent at least a quadratic extrapolation.

In the Fig. 4, the perturbed trajectory $\mathrm{tr}_{\mathrm{P2}}$ is shown for which the conditions (19) and (20) do not apparently apply, yet it is intuitively obvious that such a case can occur. The linearized relations are not sufficient to its clarification. This suggests that it is necessary to deal with problems of exact/strong norm effects in even more detail.

III. Effect And Green's Functions In Vertical Co-Ordinate Domain

The Effect and Green's functions are transformed – for practical reasons [4, 6, 8, 13 – 16, 29 – 31] – from the time domain to the vertical co-ordinate y domain. For the derivation of the fundamental relations, the appropriate procedure is designed by M. Garnier (Garnier's notation) [4, 6, 8, 14, 16], whereas in practice, the NATO countries and the former Soviet union use the Bliss' modification (Bliss' notation) [4, 6, 8, 13,15, 16, 31].

In the numerical calculations of the Green's functions g(y), we are faced with the problem of division by zero for $y = y_0$. In the subsection 3.4 we will propose a procedure how to work around the problem by using the approximation of the generalized Garnier's effect function $Q_{\mathrm{CG}}(y)$ near the point $y = y_0$. An approximation can be computed by the method of least squares.

3.1 *Definition of generalized Effect and Green's functions*

For shooting at common trajectories, measured deviations ($\Delta\mu$, $\delta\mu$) are evaluated depending on coordinate y of the projectile trajectory, thus ($\Delta\mu(y)$, $\delta\mu(y)$) is used [4, 6, 8, 10, 16, 27 –31]. For a detailed explanation see [4] – the subsection 2.5.

Therefore, it is necessary to modify the relations (1) to (6). We will use the function $t_{\mathrm{P}} = F(y)$ valid for standard trajectory. It is a one-to-two function. Such an essential failure will be eliminated by deliberating the particular dependence separately for the ascending branch (AB) $t_{\mathrm{P,AB}} = t_{\mathrm{P1}}(y) = F_{\mathrm{AB}}(y)$ and separately for the descending branch (DB) $t_{\mathrm{P,DB}} = t_{\mathrm{P2}}(y) = F_{\mathrm{DB}}(y)$. It holds that $t_{\mathrm{P1}}(y) \leq t_{\mathrm{P2}}(y)$.

To facilitate algorithm development for following calculations on a digital computer, it is convenient to define firstly the generalized Green's and effect functions in the time domain [4] – the subsection 2.5.1.

We define the generalized Green's function by the relationship

$$g_{\mathrm{C}}(t_{\mathrm{P}}) = g_{\mathrm{C}}(N_\mu, t_{\mathrm{P}}, t_{\mathrm{PI},i}) = \begin{bmatrix} 0 & t_{\mathrm{P}} < t_{\mathrm{O},i} \\ g(t_{\mathrm{P}}) & t_{\mathrm{P}} \in \langle t_{\mathrm{O},i}, t_{\mathrm{PI},i} \rangle \\ 0 & t_{\mathrm{P}} > t_{\mathrm{PI},i} \end{bmatrix}, \qquad (21)$$

where the function $g(t_{\mathrm{P}})$ is given by the relations (4) and (5). The function g_{C} is defined on the interval $(-\infty, \infty)$ and usually has a discontinuity of the first kind for the $t_{\mathrm{P}} = t_{\mathrm{O},i}$. For the $t_{\mathrm{P}} = t_{\mathrm{PI,i}}$ is usually continuous.

It applies for the generalized effect function in accordance with the relations (4) and (5)

$$Q_{\mathrm{CP}}(t_{\mathrm{P}}) = Q_{\mathrm{CP}}(N_\mu, t_{\mathrm{P}}, t_{\mathrm{PI},i}) = N_\mu \cdot \int_{t_{\mathrm{P}}}^{\infty} g_{\mathrm{C}}(t'_{\mathrm{P}}) \cdot \mathrm{d}t'_{\mathrm{P}} = \begin{bmatrix} Q_{\mathrm{P}}(t_{\mathrm{O},i}) & t_{\mathrm{P}} < t_{\mathrm{O},i} \\ Q_{\mathrm{P}}(t_{\mathrm{P}}) & t_{\mathrm{P}} \in \langle t_{\mathrm{O},i}, t_{\mathrm{PI},i} \rangle \\ 0 & t_{\mathrm{P}} > t_{\mathrm{PI},i} \end{bmatrix} . (22)$$

It applies by analogy to (5) (does not apply at points where the derivative of the function is not continuous)

$$g_{\mathrm{C}}(t_{\mathrm{P}}) = -\frac{1}{N_\mu} \cdot \frac{\mathrm{d}Q_{\mathrm{CP}}}{\mathrm{d}t_{\mathrm{P}}} = -\frac{\mathrm{d}R_{\mathrm{CP}}}{\mathrm{d}t_{\mathrm{P}}} = -\left(\frac{\sigma_{\mathrm{Q}} \cdot N_{\mathrm{Q}}}{N_\mu} \right) \cdot \frac{\mathrm{d}r_{\mathrm{C}}}{\mathrm{d}t_{\mathrm{P}}}, \qquad (23)$$

where

$R_{\mathrm{CP}}(t_{\mathrm{P}})$ is a generalized unit effect function,

$r_{\mathrm{C}}(t_{\mathrm{P}})$ is a generalized weighting factor function (curve).

Now, we define generalized effect functions for the ascending branch $Q_{\mathrm{CP,AB}}(y)$ and for the descending branch $Q_{\mathrm{CP,DB}}(y)$

$$Q_{\mathrm{CP,AB}}(y) = Q_{\mathrm{CP}}(t_{\mathrm{P1}}(y)) \wedge t_{\mathrm{P1}} \leq t_{\mathrm{S}} , \qquad (24)$$

$$Q_{\mathrm{CP,DB}}(y) = Q_{\mathrm{CP}}(t_{\mathrm{P2}}(y)) \wedge t_{\mathrm{P2}} \geq t_{\mathrm{S}} , \qquad (25)$$

where t_{S} is the moment of the projectile passage of the top $[(t, x, y, z)_{\mathrm{S}}$ and $y_{\mathrm{S}} = \max y]$ of the basic trajectory – closer the subsection 2.1 and [4] (the subsection 2.5.1). Both functions may contain points where they are not smooth.

3.2 *Garnier's notation of generalized Effect and Green's functions*

The generalized Garnier's effect function is defined on the interval $\langle y_{\min}, y_{\max} \rangle$ by the relationship [4, 6]

$$Q_{\mathrm{CG}}(y) = Q_{\mathrm{CP,AB}}(y) - Q_{\mathrm{CP,DB}}(y) . \qquad (26)$$

This function may contain points at which it is not smooth.

Generalized Garnier's notation of the Green's function $g_{\mathrm{CG}}(y)$ is given by the implicit relation

$$Q_{\mathrm{CG}}(y) = Q_{\mathrm{CG}}(N_\mu, y) = N_\mu \cdot \int_{y}^{y_{\max}} g_{\mathrm{CG}}(y') \cdot \mathrm{d}y' . \qquad (27)$$

Hence it follows (not in points where derived function is not smooth)

$$g_{\mathrm{CG}}(y) = -\frac{1}{N_\mu} \cdot \frac{\mathrm{d}Q_{\mathrm{CG}}}{\mathrm{d}y} = -\frac{\mathrm{d}R_{\mathrm{CG}}}{\mathrm{d}y} = -\left(\frac{\sigma_{\mathrm{Q}} \cdot N_{\mathrm{Q}}}{N_\mu} \right) \cdot \frac{\mathrm{d}r_{\mathrm{CG}}}{\mathrm{d}y} , \quad (28)$$

where
$R_{CG}(y)$ is the generalized Garnier's unit effect function,
$r_{CG}(y)$ is the generalized Garnier's weighting factor function (curve).
The absolute ballistic deviation/perturbation $\Delta\mu_B$ of the ballistic meteo parameter μ – in analogy to the relation (6) – is

$$\mu_B = -\int_{y_{min}}^{y_{max}} \Delta\mu(y')\cdot\left(\frac{dr_{CG}(\Delta\mu_{B0}, y)}{dy}\right)\cdot dy' = \left(\frac{N_\mu}{\sigma_Q \cdot N_Q}\right)\cdot \int_{y_{min}}^{y_{max}} g_{CG}(\Delta\mu_{B0}, y')\cdot dy' \,. \quad (29)$$

By combining relations (24) to (26) and (28), to which we add the relations $t_{P,AB} = t_{P1}(y) = F_{AB}(y)$, $t_{P,DB} = t_{P2}(y) = F_{DB}(y)$ and $dy = |v_y|\cdot dt_P$, we derive the following relation for the generalized Garnier's notation of Green's function

$$g_{CG}(y) = g_{CG,AB}(y) + g_{CG,DB}(y), \quad (30)$$

where

$$g_{CG,AB}(y) = \frac{g_C(t_{P1}(y))}{|v_y(t_{P1}(y))|} = -\frac{1}{N_\mu}\frac{dQ_{CP,AB}(y)}{dy}$$

is the component corresponding to the ascending branch and

$$g_{CG,DB}(y) = \frac{g_C(t_{P2}(y))}{|v_y(t_{P2}(y))|} = +\frac{1}{N_\mu}\frac{dQ_{CP,DB}(y)}{dy}$$

is the component corresponding to the descendingbranch. Both components may contain points of discontinuity of the first kind.

If for some y_0 the vertical component of ground speed is $v_y = 0$, then for $y = y_0$ the Green's function diverges ($g_{CG}(y_0) \to +/-\infty$). Limitations of the influence of this complication on the numerical calculations will be discussed in the subsection 3.4.

3.3 ***Bliss' notation of generalized Effect and Green's functions***

The generalized Bliss' effect function is defined on the interval $\langle y_{min}, y_{max}\rangle$ by the relation [4, 6]

$$Q_{CB}(y) = Q_{CG}(y_{min}) - Q_{CG}(y) \,. \quad (31)$$

The generalized Bliss' notation of Green function is then given by the relation

$$Q_{CB}(y) = Q_{CB}(N_\mu, y) = -N_\mu \cdot \int_{y_{min}}^{y} g_{CB}(y')\cdot dy' \,. \quad (32)$$

It follows from the relations (31) and (32) (it does not apply at points where derived function is not smooth)

$$g_{CB}(y) = -\frac{1}{N_\mu}\cdot\frac{dQ_{CB}}{dy} = -\frac{dR_{CB}}{dy} = -\left(\frac{\sigma_Q \cdot N_Q}{N_\mu}\right)\cdot\frac{dr_{CB}}{dy} = -g_{CG}(y) \,, \quad (33)$$

where
$R_{CB}(y)$ is the generalized Bliss' unit effect function,
$r_{CB}(y)$ is the generalized Bliss' weighting factor function (curve).
The absolute ballistic deviation/perturbation $\Delta\mu_B$ of the ballistic meteo parameter μ – in analogy to the relations (6) and (29) – is

$$\Delta\mu_B = +\int_{y_{min}}^{y_{max}} \Delta\mu(y')\left(\frac{dr_{CB}(\Delta\mu_{B0}, y)}{dy}\right) dy' = -\left(\frac{N_\mu}{\sigma_Q \cdot N_Q}\right)\cdot \int_{y_{min}}^{y_{max}} g_{CB}(\Delta\mu_{B0}, y')\cdot dy' \,. \quad (34)$$

3.4 ***Problem of divergence of Garnier's and Bliss' notation of generalized Green's functions***

At the end of the subsection 3.2, we pointed out the fact that the generalized Green's function in Garnier's notation $g_{CG}(y)$ may at some points diverge. It implies from the relation (33) that this problem applies also to the Bliss‘ notation$g_{CB}(y)$. We analyze the problem in this subsection and propose a solution that will allow a circumventing of the numerical difficulties associated with the mentioned singularities of the function.

First, we need to define the conditions under which divergence occurs. The divergence occurs, if the vertical component v_y of the ground speed of the projectile $\boldsymbol{v}$ (see Comment to the Table 2) is zero ($v_y = 0$), or equivalently, if the angle θ is equal to zero ($\theta = 0$).

In the 1st trajectory (i = 1, the subsection 2.1) is always $v_y > 0$.
In the 2nd and the 3rd trajectory (i = 2, 3) is $v_y = 0$ at top of the trajectory, which is identical with the top of the basic trajectory, i.e. at the time t_S,$t_{Sr} = t_S - t_{O,i}$, Fig. 1. In this case, both components $g_{CG,AB}(y)$, $g_{CG,DB}(y)$ diverge and therefore also $g_{CG}(y)$ for $y = y_{max} = y_S - y(t_{O,i}) = y_{Sr}$, so, at the end of the interval $\langle y_{min}, y_{max}\rangle$.

In the 4th trajectory (i = 4) can be $v_y = 0$ only at the beginning of the trajectory $t = t_{O,4}$, $t_r = t - t_{O,4} = 0$ and only under the condition that the angle of departure θ_0 is exactly equal to zero ($\theta_0 = 0$). In this case, it will be exactly valid $t_{O,4} = t_S$. Simultaneously, it is valid $y_{max} = y_S - y(t_{O,i}) = 0$ and $y_{min} = y_{PI,4} - y(t_{O,4}) = y_{PI,4} - y_S < 0$. Also it is $g_{CG,AB}(y) = 0$, so the $g_{CG}(y) = g_{CG,DB}(y)$ and diverges again $y = y_{max} = 0$. This variant will be referred to as the "special 4th trajectory" for short.

It follows from the relation (30) that the function $g_{CG}(y)$ is given by the quotient of two functions, so we should apply L'Hospital's rule for the analysis of its divergence. However, we will proceed differently. We create the Taylor series for the function $g_C(t_P)$ in the neighborhood of $t_P = t_S$ ($t_{Sr} = t_S - t_{O,i}$, i = 2, 3, 4) and at the same time, we can find an approximation of the course of the vertical component of the speed v_y also for the neighborhood of the top of the basic trajectory ($y_S = y(t_S)$).

First, we introduce the relative time by the relationship

$$\tau = t - t_s = t_r - t_{Sr} , \quad (35)$$

so, for the ascending branch is $\tau \le 0$ and for the descending branch is $\tau \ge 0$.

We will expand the generalized Green’s function into the Taylor/McLaurent series in the neighborhood of the point $\tau = 0$ and we will leave the first three members of the development

$$g_C(\tau) \approx g_C(0) + \dot{g}_C(0) \cdot \tau + \frac{1}{2}\ddot{g}_C(0) \cdot \tau^2 + \ldots \; . \qquad (36)$$

Furthermore, we choose a suitable approximation of the course of $v_y(\tau)$ in the neighborhood of $\tau = 0$. We assume a linear course of the vertical acceleration

$$a_y \approx -g + j_y \cdot \tau = -g \cdot (1 - \omega \cdot \tau), \qquad (37)$$

where
$g = g(y_{max} + h_G) > 0$ is the acceleration due to gravity at the top of the trajectory [ms^{-2}],
h_G is the altitude of the origin of the ballistic coordinate system (x, y, z) [m]
$j_y > 0$ is the jerk, which approximately expresses the effect of the aerodynamic drag [ms^{-3}],
$\omega = j_y/g$ approximation constant, its value is near zero $\omega \approx 0$ [s^{-1}].

The vertical component of the velocity is

$$v_y \approx -g \cdot \tau + \frac{1}{2}\omega \cdot g \cdot \tau^2 = -g \cdot \tau \cdot \left[1 - \frac{\omega}{2}\tau\right] . \qquad (38)$$

The fall in the trajectory is

$$\Delta y \approx -\frac{g}{2}\tau^2 + \frac{g \;\; \omega}{6}\tau^3 \le 0 \; . \qquad (39)$$

We introduce formally

$$\Delta y = \frac{-g}{2}\tau_0^2 = y - y_{max} = -y_{max} \cdot (1 - \xi) \le 0 \; , \qquad (40)$$

where $\xi = y/y_{max} \ge 0$ and for y_{max} we choose for the 2nd and the 3rd trajectories y_{Sr} and the special 4th trajectory abs(y_{min}). Therefore, it will be in our considerations $\xi \approx 1$. So, it is true

$$\tau_0 = T_0 \; \sqrt{[1 - \xi]} \; , \qquad (41)$$

where $T_0 = \sqrt{\dfrac{2 y_{max}}{g}}$ is the time constant [s].

By combination of the relations (39) and (40) we receive the cubic equation

$$\tau^3 - \frac{3}{\omega}\tau^2 + \frac{3}{\omega}\tau_0^2 \approx 0 \ . \qquad (42)$$

From three roots of the equation (42), the following two roots are convenient
– for the ascending branch

$$\tau_1 \approx -\tau_0 + \Delta\tau \approx -\tau_0 + \frac{\omega}{6}\tau_0^2 \le 0, \qquad (43)$$

– for the descending branch

$$\tau_2 \approx +\tau_0 + \Delta\tau \approx +\tau_0 + \frac{\omega}{6}\tau_0^2 \ge 0 \ . \qquad (44)$$

Because

$$\Delta\tau \approx \frac{1}{\omega}\cdot\left[1-\sqrt{\left[1-\frac{1}{3}(\omega\cdot\tau_0)^2\right]}\right] \approx \frac{\omega}{6}\cdot\tau_0^2 \ .$$

We substitute step by step the roots of (43) and (44) into (39) and after adjustments we obtain
– for the ascending branch

$$\left|v_{y,\mathrm{AB}}\right| \approx g\cdot\tau_0\cdot\left[1+\frac{\omega}{2}\tau_0\right], \qquad (45)$$

– for the descending branch

$$\left|v_{y,\mathrm{DB}}\right| \approx g\cdot\tau_0\cdot\left[1-\frac{\omega}{2}\tau_0\right] \ .$$

We substitute step by step the expressions (41) and (43) to (45) into the relation (36) and after adjustments we obtain the expressions
– for the ascending branch

$$g_{\mathrm{CG,AB}}(\xi) \approx \frac{g_0}{\sqrt{[1-\xi]}} - g_1 + g_2\cdot\sqrt{[1-\xi]}, \qquad (46)$$

– for the descending branch, that is simultaneously the resulting relationship for the generalized Green's function in Garnier’s notation for the special 4th trajectory

$$g_{\mathrm{CG,DB}}(\xi) = g_{\mathrm{CG,4}}(\xi) \approx \frac{g_0}{\sqrt{[1-\xi]}} + g_1 + g_2\cdot\sqrt{[1-\xi]} \ , \qquad (47)$$

– for the generalized Green’s function in Garnier’s notation for the 2^{nd} and the 3^{rd} trajectories in the accordance with the relationship (30)

$$g_{\mathrm{CG,2,3}}(\xi) \approx \frac{2\cdot g_0}{\sqrt{[1-\xi]}} + 2\cdot g_2\cdot\sqrt{[1-\xi]}, \qquad (48)$$

where

$$g_0 \approx \frac{g_C(0)}{g\cdot T_0} \ ,$$

$$g_1 \approx \frac{1}{g}\cdot\left[\dot{g}_C(0) + \frac{\omega}{2}g_C(0)\right],$$

$$g_2 \approx \frac{T_0}{g}\cdot\left[\frac{1}{2}\ddot{g}_C(0) + \frac{2\omega}{3}\dot{g}_C(0)\right].$$

We associate the expressions (47) and (48) into one relationship

$$g_{CG}(\xi) \approx \frac{a_0}{\sqrt{[1-\xi]}} + a_1 + a_2\cdot\sqrt{[1-\xi]}, \quad (49)$$

where
$a_0 = g_0{\cdot}k_0$, $a_1 = g_1{\cdot}k_1$, $a_2 = g_2{\cdot}k_2$

and it holds

– for the 2nd and the 3rd trajectories $k_0 = k_2 = 2$, $k_1 = 0$,

– for the special 4th trajectory $k_0 = k_1 = k_2 = 1$.

It follows from the relations (33), (46) to (49) that the divergence of the generalized Green's functions in Garnier's and Bliss' notations occurs for $y = y_{\max}$, ($\xi = 1$), if $g_1 \neq 0$, so if $g_C(\tau) \neq 0$ for $\tau = 0$. Then

– in the case of $g_C(0) > 0$, it is valid $g_{CG}(0) \to +\infty$; this is the most common case when we calculate a positive perturbation,

– in the case of $g_C(0) < 0$, it is valid $g_{CG}(0) \to -\infty$.

If $g_C(\tau) = 0$ for $\tau = 0$, then $\mathrm{abs}(g_{CG}(0)) < \infty$. The function $g_C(t_P)$ is in the most cases calculated numerically, and so due to the influence of calculation errors, this alternative is mostly numerically unstable.

In accordance with the relationship (27), we perform the integration of the relation (49) and after adjustment we obtain

$$Q_{CG}(\xi) \approx b_0 \cdot \sqrt{[1-\xi]} + b_1 \cdot (1-\xi) + b_2 \cdot \sqrt{[1-\xi]^3} \ , \qquad (50)$$

where

$b_0 \approx 2 \cdot y_{\max} \cdot N_\mu \cdot a_0$,

$b_1 \approx y_{\max} \cdot N_\mu \cdot a_1$,

$b_2 \approx \dfrac{2}{3} y_{\max} \cdot N_\mu \cdot a_2$.

If the Garnier's algorithm is used for the calculation of the effect functions $Q_P(t_P)$, then the values of the generalized Garnier's effect function, respectively $Q_{CG}(y)$ and $Q_{CG}(\xi)$, can be immediately simply counted. In this case, we use the relationship (50) as the approximation formula and we approximate the numerically calculated course of $Q_{CG}(\xi)$ for $\xi \to 1$. In the case of the 2nd or 3rd trajectories, we put a priori $b_1 = 0$. We can use two/three points approximation, or the method of least squares. So, we get estimates of the coefficients b_j, $j = 0, 1, 2$.

Subsequently, we compute from the coefficients b_j the estimates of the coefficients a_j, $j = 0, 1, 2$, and from them it is possible to determine estimates of the coefficients g_j, $j = 0, 1, 2$.

We obtain the course of the generalized Green's function in Garnier's notation $g_{CG}(y)$ using the numerical derivative of the corresponding generalized effect function $Q_{CG}(y)$ in accordance with (28). Only for values y close to the value $y_{\max}$, we use for calculations the approximation relationship (49). Thus, we circumvent the above-analyzed problem associated with the divergence of the function $g_{CG}(y)$.

IV. Modification Of Selected Functions

In this section we will use only the Bliss' notation of generalized effect functions $Q_{CB}(y)$ and Green's functions $g_{CB}(y)$, as it is usual in NATO countries [13, 15, 29, 31] and countries of the former Soviet bloc [8, 10]. Thus, we can skip the abbreviation "CB".

In all the expressions used for the practical calculations, the generalized Green's functions $g(y)$ occur, or the first derivatives of generalized weighting factor functions (curves) WFFs $r'(y)$, which are the normed Green's functions – relationship (33). But this means that in using them we will be forced to solve problems with their divergence, which we discussed in the subsection 3.4.

Our goal is to adjust these relationships, to use – instead of the generalized Green's functions and the first derivative of WFFs – the generalized WFFs $r(y)$, which are the normed generalized effect functions $Q(y)$ – the relationship (33). This completely avoids the problems associated with the divergence of the generalized Green's functions.

4.1 *Modification of Reference height of trajectory formula*

In the article [6], we have derived the relation (28) for the normed moments of the first derivative of the weighting function WFF $r(N_\mu,\eta)$, so the normed Green's function

$$m_{WFFi} = \int_0^1 \eta^i \ \frac{\mathrm{d}r(N_\mu,\eta)}{\mathrm{d}\eta} \cdot \mathrm{d}\eta,\ i = 0,1,\dots \ , \qquad (51)$$

where

$$\eta = \frac{y - y_{\min}}{y_{\max} - y_{\min}} = \frac{h - h_{\min}}{h_{\max} - h_{\min}} = \frac{\Delta h}{\Delta h_m} \ ,$$

where

$h = h_G + y$ is the altitude corresponding to the vertical coordinate of the y,

h_G is the altitude of the horizontal plane (x, z), $y = 0$ of the ballistic system,
$\Delta h = h - h_{\min} = y - y_{\min}$ and
$\Delta h_m = h_{\max} - h_{\min} = y_{\max} - y_{\min}$.

For the modification of the relation (51) (see [6], the relationship (28)), we use the integration by parts, whereby we come out of a relationship

$$\left(\eta^i \cdot r\right)' = i \cdot \eta^{i-1} \cdot r + \eta^i \cdot r' \quad . \tag{52}$$

After performing the integration and adjustment, we get the new relationship

$$m_{\mathrm{WFF},i} = r(1) - M_{\mathrm{WFF},i}\,,\ i = 0,1,\ldots, \tag{53}$$

where

$$M_{\mathrm{WFF},i} = i\int_0^1 \eta^{i-1} \cdot r\left(N_\mu, \eta\right) \cdot \mathrm{d}\eta,\ i = 0,1,\ldots, \tag{54}$$

where
$r(1) = r_N(N_\mu, \eta)$ and $\eta = 1$ – value of WFF, $r(1) \in \langle 0, 1\rangle$. In the case of the traditional normalization, it holds always $r(1) = 1$, more details in [6].

It is true for the first two moments

$$m_{\mathrm{WFF},0} = r(1)\ , \tag{55}$$

$$m_{WFF,1} = r(1) - S\ , \tag{56}$$

where

$$S = S_{WFF} = \int_0^1 r\left(N_\mu, \eta\right)\ \mathrm{d}\eta\ . \tag{57}$$

In [6] we have derived the relation (39) for the coefficient of the generalized reference height K_{CR}. In this relation, we substitute the relation (56) and thus we obtain a new expression

$$K_{CR} = \eta_{CR} \approx 2 \cdot \left[\left(r(1) - S\right) - \left(1 - r(1)\right) \cdot \left(\frac{a_0}{a_1}\right)\right]\ . \tag{58}$$

In the case of the traditional normalization (r(1) = 1), it will be

$$K_{CR} = \eta_{CR} \approx 2\ \left(1 - S\right)\ . \tag{59}$$

This relation has been hitherto derived only semi-empirically [8]. We have reported its derivation in [1] – the relation (28) and again in [6] – the relation (66). Now we have presented its correct derivation, from which it follows that the relation (59) is valid quite generally, but only when using traditional normalization ($r(1) = 1$). At the same time, we have derived the relation (58), which has a completely universal validity, which is our next asset to the theory of reference height of projectile trajectory (RHT).

It holds universally for the generalized reference height ([6] – the relation (40))

$$Y_{CR} = \Delta h_{CR} = K_{CR} \cdot \Delta h_m\ . \tag{60}$$

The clarification of the whole problem of the RHT can be found in [6].

4.2 *Modification of absolute and relative ballistic perturbation formulas*

It is necessary to modify the expression for the absolute ballistic deviation/perturbation $\Delta\mu_B$ of the ballistic meteo parameter μ, which is given by the relation (34). The same procedure can be applied for the modification of the relations (6) and (29).

For the modification of the relation (34), we use the integration by parts and the following relation (as the initial one)

$$\left(\Delta\mu \cdot r\right)' = \Delta\mu' \cdot r + \Delta\mu \cdot r'\ . \tag{61}$$

After performing the integration and adjustment we get a new relationship

$$\Delta\mu_{\mathrm{B}} = \Delta\mu(1)\cdot r(1) - \int_0^1 \left(\frac{d\Delta\mu(\eta)}{d\eta}\right)\cdot r_{\mathrm{CB}}(\Delta\mu_{\mathrm{B0}}, \eta')\cdot \mathrm{d}\eta' . \quad (62)$$

The course of $\Delta\mu(y)$ is usually obtained by a measurement, therefore, we recommend to smooth it by the filtration in an appropriate way.

One of the methods of filtration is averaging [6], and for this purpose we have derived the relation (53) analogous to the above relation (34), in which, however, the average size of the quantity $\Delta\mu(y)$ is used

$$\Delta\mu_{\mathrm{AV}}(y) = \frac{1}{y - y_{\min}} \int_{y_{\min}}^{y} \Delta\mu(y)\,\mathrm{d}y = \frac{1}{\eta}\int_0^{\eta} \Delta\mu(\eta)\,\mathrm{d}\eta \; . \quad (63)$$

In relation (53) of [6], the second derivative of WFF r’’(t) performs, therefore we replace this relationship by a new one. We go out of the relationship (34). We use for the derivation the following relationship, which was originated by the differentiation of the relation (63)

$$\eta \cdot \frac{\mathrm{d}\Delta\mu_{\mathrm{AV}}(\eta)}{\mathrm{d}\eta} = \Delta\mu(\eta) - \Delta\mu_{\mathrm{AV}}(\eta) \; . \quad (64)$$

We use repeatedly the integration by parts formula, using relations

$$(\Delta\mu_{\mathrm{AV}} \cdot r)' = \Delta\mu'_{\mathrm{AV}} \cdot r + \Delta\mu_{\mathrm{AV}} \cdot r' \quad (65)$$

and

$$(\eta \cdot \Delta\mu'_{\mathrm{AV}} \cdot r)' = \Delta\mu'_{\mathrm{AV}} \cdot r + \eta \cdot \Delta\mu''_{\mathrm{AV}} \cdot r + \eta \cdot \Delta\mu'_{\mathrm{AV}} \cdot r' \; . \quad (66)$$

We use gradually the relations (64) to (66) with the relationship (34) and after adjustments we obtain the final relationship for the absolute ballistic deviation/perturbation $\Delta\mu_{\mathrm{B}}$ of the ballistic meteo parameter μ

$$\Delta\mu_{\mathrm{B}} = \mu_{\mathrm{B}}(1) - \int_0^1 \left(\frac{\mathrm{d}\Delta\mu_{\mathrm{AV}}(\eta)}{\mathrm{d}\eta}\right)_{\mathrm{red}} \cdot r_{\mathrm{CB}}(\Delta\mu_{\mathrm{B0}}, \eta')\cdot \mathrm{d}\eta', \quad (67)$$

where

$$\mu_{\mathrm{B}}(1) = \left[\Delta\mu_{\mathrm{AV}}(\eta) + \frac{\mathrm{d}\Delta\mu_{\mathrm{AV}}(\eta)}{d\eta}\right]\cdot r(\eta) \text{ for } \eta = 1, \quad (68)$$

$$\left(\frac{\mathrm{d}\Delta\mu_{\mathrm{AV}}(\eta)}{\mathrm{d}\eta}\right)_{\mathrm{red}} = 2 \cdot \frac{\mathrm{d}\Delta\mu_{\mathrm{AV}}(\mu)}{\mathrm{d}\eta} + \eta \cdot \frac{\mathrm{d}^2\Delta\mu_{\mathrm{AV}}(\mu)}{\mathrm{d}\eta^2} \; .(69)$$

The procedures used in this section can be applied by analogy to other relationships.

V. Conclusion

This article concludes the basics of the improved theory of generalized effect and Green’s functions, as special sensitivity functions.

This theory allows performing an effective sensitivity analysis of the properties of non-standard projectile trajectories. The theory is fully linked to the more general theory of sensitivity analysis of dynamical systems, so the results can be interpreted in a broader context.

In the following period, we will deal with applications of this theory to solutions of partial problems occurring in the meteo-ballistics.

Acknowledgments

This work originated with the support of financing from the Research Project for the Development of the Department of Weapons and Ammunition, Faculty of Military Technology, University of Defence, Brno, DZRO VYZBROJ.

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