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Using The Energy Method To Determine Large Deflections Of Flat Textile Structures

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Abstract:
The article describes how to solve alternative differential equations of heavy elastica using the energy method. These may be used for the solution of large deflection problems of flexible strips such as fabric. This study is based on the minimum energy principle. Potential and bending energy are given in a general form. The minimum energy solution was based on the calculus of variations. The use of the heavy elastica differential equation is well accepted and several approximations for special cases have been discussed in the literature. The results obtained from the analysis can be used, for example, to simulate the free folding of flat textiles as well as the bending of linear structures.
Keywords: *energy methods; fabric; heavy elastica; numerical methods; potential energy.*

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I. Introduction

In the problems of textile mechanics, large deflections are very often taken into account, as well as stability problems. Due to the fact that the mechanics of textiles is a relatively young field of science, many problems occurring in it are solved on the basis of classical mechanics, the theory of elasticity, etc. In addition, classical theory of “elastica” is often used to analyze linear and flat textile structures. The above-mentioned issues are the main subject of this article.

Deflection of beams or plates has been the subject of numerous engineering problems nowadays which have very attractive civilian applications. Many examples of a small deflection theory of a beams subjected to a vertical concentrated force at the free end can be found in many mechanics textbooks, e.g. Gere [1]. However, in the small deflection theory, the principle of superposition with small angle assumptions is valid and the equation of deflection shows proportionality between the deflection and the externally applied force. On the other hand, when deflections are large, the small angle assumption and the principle of superposition are no longer valid, the problem becomes increasingly difficult and the analytical solution for many loading conditions does not exist due to the presence of a nonlinear term in the governing equation. Lee et al. [2] investigated large deflection of a linear elastic cantilever beam of variable cross section under combined loading by means of the Runge-Kutta method. Baker [3] obtained large deflection profiles of linear elastic tapered cantilever beams under arbitrary distributed loads by means of a weighted residual solution of the Bernoulli-Euler bending moment equation. Dado and AL-Sadder [4] presented a new technique for large deflection analysis of non-prismatic cantilever beams based on the integrated least square error of the non-linear governing differential equation in which the angle of rotation is represented by a polynomial. Shatnawi and AL Sadder [5] studied exact large deflection of non-prismatic, nonlinear bi-modulus cantilever beams subjected to a tip moment by applying a power series approach to analytically solve highly nonlinear simultaneous first order differential equations. Shvartsman [6] examined large deflections of a cantilever beam subjected to a follower force by reducing a nonlinear two-point boundary-value problem to an initial-value problem by change of variables, then solving without iterations. Szablewski and Kobza [7] developed the numerical analysis of a test for the bending rigidity of textiles as proposed by Peirce. The mathematical model treated textile product as an elastica which is subject to large deflections. AL-Sadder and AL-Rawi [8] developed quasi linearization finite differences for large-deflection analysis of non-prismatic slender cantilever beams subjected to various types of continuous and discontinuous external variable distributed and concentrated loads in horizontal and vertical global directions. Ibrahimbegovic [9] studied large displacement of beams by implementing finite element analysis to three-dimensional finite-strain Reissner beam theory, where beam element reference axes are represented by arbitrary space-curved lines. Cantilever beams of non linear materials have also been studied. Lewis and Monasa [10] numerically studied large deflections of cantilever beams made of non-linear materials subjected to one vertical concentrated load at the free end using a fourth order Runge-Kutta method. Lee [11] examined large deflection of cantilever beams of non-linear elastic material under the effects of combined loading by using Butcher’s fifth order Runge Kutta method. Rezazadeh [12] developed a comprehensive model to study nonlinear behavior of multilayered micro beam switches for the application of micro-electromechanical mechanical systems (MEMS), in which the derived nonlinear equation was numerically solved using the nonlinear finite difference method. Antman [13] studied large lateral buckling of nonlinearly elastic beams subjected to flexure, torsion, extension or shear. This configuration is described by a position vector function and an orthonormal pair of vector functions of a real variable which is interpreted as a scaled arc length parameter of the straight line of centroids of a beam in its natural reference configuration. Cesnik et al. [14] presented a refined theory of composite beams. The basis for the theory is the variational-asymptotic method, a mathematical technique by which the three-dimensional analysis of composite beam deformation can be split into a linear, two-dimensional, cross-sectional analysis and a nonlinear, one-dimensional beam analysis. An original method of examining the state of equilibrium of a flat textile structure was presented by Szablewski [15]. This kind of structure was modeled as an inextensible elastica loaded with its own weight and axial force.

Large deflection of cantilever beams that are prismatic and made of linear elastic material have been the subject of numerous studies in which the beam is subjected to a uniformly distributed load. Seames and Conway [16] presented a numerical method for calculating large deflections of cantilever beams under uniform loading. This numerical method assumed that the elastic axis of the beam could be approximated by a number of circular arcs tangent to one another at their points of intersection, using the Bernoulli-Euler equation to determine the radius of each circular arc. Rohde [17] obtained an approximate solution for the large deflection of a cantilever beam subject to a uniformly distributed load by expanding the slope in a power series of the arc length. Lee et al. [18] analyzed stresses and displacements experimentally in largely deflected cantilever beams subjected to uniformly distributed loads by means of photoelasticity. This analysis demonstrated that for the case of a beam material having a small modulus of elasticity value with gravity acting alone as a uniform load that large deflections would occur. Szablewski and Korycki [19] presented a general theory concerning shape determination of coplanar elastica subjected to static bending by means of displacements. The displacements were described by the coordinates of points of initially unbending elastica and the loads imposed. Belendez et al. [20] analyzed large deflections of a uniform cantilever beam under the action of a combined load consisting of a uniformly distributed load and an external vertical concentrated load applied at the free end. This analysis obtained a numerical solution using an algorithm based on the Runge-Kutta-Felhberg method and compared the numerical results with experimental results. Frisch Fay [21] solved for the large deflection of a cantilever beam under two concentrated loads in terms of elliptic integrals. Bisshopp and Drucker [22] solved for the large deflection of a cantilever beam subjected to one concentrated load, acting vertically downward at the free end of the beam, also in terms of elliptic integrals. The work of [21] and [22] are all based on the fundamental Bernoulli-Euler theorem which states that the curvature is proportional to the bending moment. Similar procedure applying Jacobi elliptic integrals of first and second types by considering only end-load is used by Howell and Midha [23].

A uniform cantilever beam under the action of a combined load consisting of a uniformly distributed load and an external vertical concentrated load applied at the free end were analyzed by Belendez et al. [24]. They tried to find an exact analytical solution, however upon discovering that one does not exist, proceeded to apply a mixed numerical and analytical approach to solve for the deflected shapes. Saxena and Kramer [25] proposed a numerical integration scheme that requires special consideration for the occurrence of any inflection point within the beam for combined end loading.

In the article presented below the alternative differential equations of heavy elastica using the energy method were described. The results obtained from the analysis can be used to simulate the free folding of flat textiles as well as the bending of linear structures.

II. Material And Methods

Materials

In this paper, it is assumed that during the run of bending effect the flat strip of the fabric will be represented as its longitudinal section. The mathematical model will be described as a flat deflection curve, i.e. heavy elastica as shown in Figure 1. It is assumed that the particular longitudinal sections do not act on each other by internal forces (plane stress). Furthermore, the constancy of properties along the whole width of the bending strip is assumed. Therefore, instead of studying the strip of fabric, the numerical analysis will be concerned with deflections of heavy elastica of a given bending stiffness B and appropriate weight per unit area w. Furthermore, it will be assumed that the elastica is non-inextensible. It should also be pointed out that the assumption of inextensibility is somewhat limiting. However, this assumption is often made in large-deflection analysis. Each point of the centre line of elastica is defined by a curvilinear coordinate s measured along the elastica. $x(s)$, $y(s)$ are functions of the variable s in a fixed coordinate system.

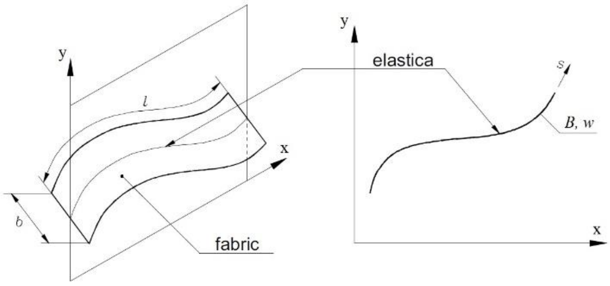

Fig. 1 The model of fabric approximated by elastica

Definition of Variables

A piece of strip of length L and width b is clamped as shown in Figure 2. For a strip with constant width b, the width does not play any role in further calculations; thus it is only necessary to consider the x-y plane. As will be seen later, it is convenient to define a general bending curve according to Figure 2.

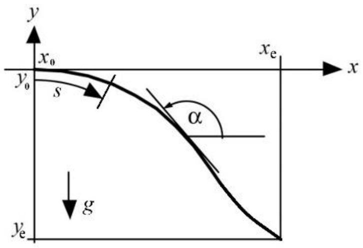

Fig. 2 Definition of variables

The curve starts at the point (x_0, y_0) and ends at (x_e, y_e). The arc length is calculated from x_0, α is the slope angle and g the gravity. At $y = 0$ the potential energy is defined to be zero.

Definition of Energy Equations

For a linear relationship between the curvature k and the bending moment M, the total energy of potential and bending energy for a strip of unit width is then given by

$$E_{tot} = \int_0^L \left[w\,y(s) + \frac{1}{2} B k^2(s) \right] ds \tag{1}$$

where w is the weight per unit area, $k(s)$ is the curvature and B the bending stiffness per unit width which is equivalent to the product of modulus of elasticity times the moment of inertia ($E{\cdot}I$). The calculus of variation will be used to solve the problem. In order to avoid formulating the isoperimetric problem related to the additional constraint of the given length of elastica, the calculus of variations will be applied directly to the functional (1). The curvature $k(s)$ can be defined as below. Since $dy/ds = \sin(\alpha)$ therefore $\alpha = \arcsin(dy/ds)$. Hence

$$k(s) = \frac{d\alpha}{ds} = \frac{d}{ds}\left(\arcsin(dy/ds)\right) = \frac{d^2y/ds^2}{\sqrt{1-(dy/ds)^2}} \tag{2}$$

$$k(s) = \frac{y''(s)}{\sqrt{1-y'^2(s)}} \tag{3}$$

$$E_{tot} = \int_0^L \left[w\,y + \frac{B}{2}\frac{y''^2}{(1-y'^2)} \right] ds \tag{4}$$

The problem is therefore to find the minimum of the functional $J[y]$ where

$$J[y] = \int_0^L F(s, y, y'')ds \tag{5}$$

$$F(s,y,y'') = w\,y + \frac{B}{2}\frac{y''^2}{(1-y'^2)} \tag{6}$$

Method of Solving Energy Equations

The problem is to find a function $y(s)$ which satisfies the following conditions: the total energy must be minimised with the condition that the length from the starting point to the end point equals the given length L. Since the functional $J[y]$ depends on derivatives of a higher-order than one, the Euler-Poisson equation should be used as a condition for the existence of an extremum.

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \tag{7}$$

Details on the calculus of variation can be found, among others, in the work [26].
Applying the calculus of variations leads to the implicit differential equation for the bent shape of strips.

$$0 = \frac{w}{B} + \frac{y''''}{1-y'^2} + \frac{4y'y''y''' + y''^3}{(1-y'^2)^2} + \frac{4y'^2y''^3}{(1-y'^2)^3} \tag{8}$$

Equation 8 is the final differential equation which describes the bent shape of strips with constant bending stiffness B and weight per unit area w. The results of calculations will be presented below.

III. Result

Numerical results were obtained using the fourth-order Runge-Kutta method for the cantilever beam. The beam was fixed at one end and free at the other end.

Standard Equations for Cantilever Beam

The standard Equation 9 describing fabric bending behavior.

$$\frac{d\alpha}{ds} = \frac{M}{B} \qquad \frac{dM}{ds} = q(s-L)\cos\alpha$$
$$\frac{dx}{ds} = \cos\alpha \qquad \frac{dy}{ds} = \sin\alpha \tag{9}$$

where M is the bending moment, B is the bending stiffness and q is the linear weight ($q = w/b$).
B is often replaced by the so-called bending length c according to the following relationship.

$$B = c^3 w \quad \text{or} \quad c = \sqrt[3]{B/w} \tag{10}$$

The fabric approximated by an elastica is fixed as in Peirce's cantilever test Figure 3.

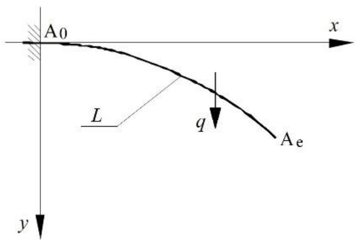

Fig. 3 Elastica in fixed coordinate system

More details can be found in the work [7].

Energy Equation in s-y Space

From the Equation 8 the energy-model is given by the following differential equation.

$$y'''' = -\frac{w(1-y'^2)}{B} - \frac{4y'y''y''' + y''^3}{1-y'^2} - \frac{4y'^2y''^3}{(1-y'^2)^2} \tag{11}$$

The initial conditions at the free end for the second and third derivative of y should be set to zero: $y^{(2)}(L) = y^{(3)}(L) = 0$. The value $y(0)$ can be chosen arbitrarily, because the bent shape of a heavy elastica is independent of the height of the starting point $y(0)$.

Results and Discussion

The dimensionless parameter *W*, defined by Equation (12), was used in the calculations.

$$\lambda = \frac{wL^3}{B} \tag{12}$$

Therefore, *w*/*B* in Equation 11 can be replaced by

$$\frac{w}{B} = \frac{\lambda}{L^3} \tag{13}$$

Calculations were performed for the three data sets.

1) $B_1 = 2.5\,\text{Nm}$, $w_1 = 2.5\,\text{N/m}^2$, $\lambda_1 = 1$
2) $B_2 = 0.3125\text{Nm}$, $w_2 = 2.5\,\text{N/m}^2$, $\lambda_2 = 8$
3) $B_3 = 0.2083\text{Nm}$, $w_3 = 2.5\,\text{N/m}^2$, $\lambda_3 = 12$

In all three cases L = 1 m and the corresponding Young's modulus: E_1 = 30 GPa, E_2 = 3.75 GPa, E_3 = 2.5 GPa.

Figure 4 shows the numerical results of the differential equation for one end fixed horizontally and the other end free to bend under its own weight for all three data sets. The Figure presents an overlay of results of two considered models: standard-model from Equation 9 and energy-model from Equation 11. As expected, all results are identical.

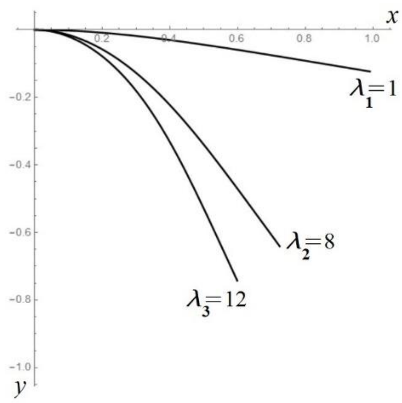

Fig. 4 The shape of the bent elastica for three cases

For small values the energy solutions show better numerical convergence than the standard approach. For bigger values the energy approach shows the best performance of convergence over the other approach.

In order to better verify the results, a bending simulation was performed using the finite element method (FEM).

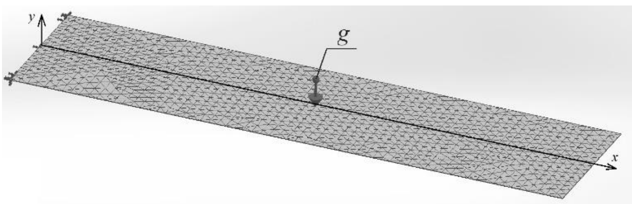

Fig. 5 The FEM model of fabric strip

Figure 5 presented the FEM-model for only one case:
$B = 2.5\,\text{Nm}, \quad w = 2.5\,\text{N/m}^2, \quad \lambda = 1,$
and the bent shape is shown in Figure 6.

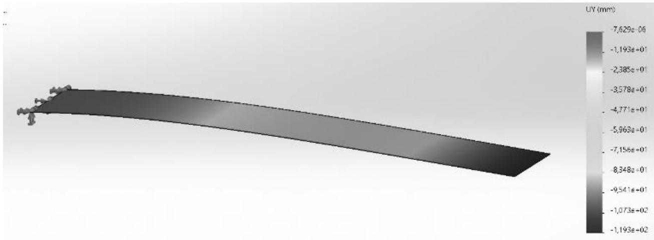

Fig. 6 The result for the FEM model

The deflection of the free end for the FEM-model is $y(L)_{\mathrm{FEM}}$ = 119 mm. For comparison, the deflection of the free end for the energy-model was obtained with a value of $y(L)_{\mathrm{Energy}}$ = 123.5 mm. $y(L)$ for the FEM-model is 3.6 percent smaller than for the energy-model. The agreement of the results is satisfactory.

An approximation to the energy differential equation was given also for horizontally clamped fabrics by Grießer and Taylor [27]. The new differential equation allow an entirely new field for the exploration of approximations for the shape of bent flexible strips.

IV. Conclusion

This article described the new form of differential equation for the shape of bent flexible strips. Equations for potential energy and bending energy are considered. The shape which is given by the minimum energy solution was found using the calculus of variations. The solution for energy-model were compared with the standard-model. The new form of differential equation will allow an entirely new field of approximation to be explored.

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